

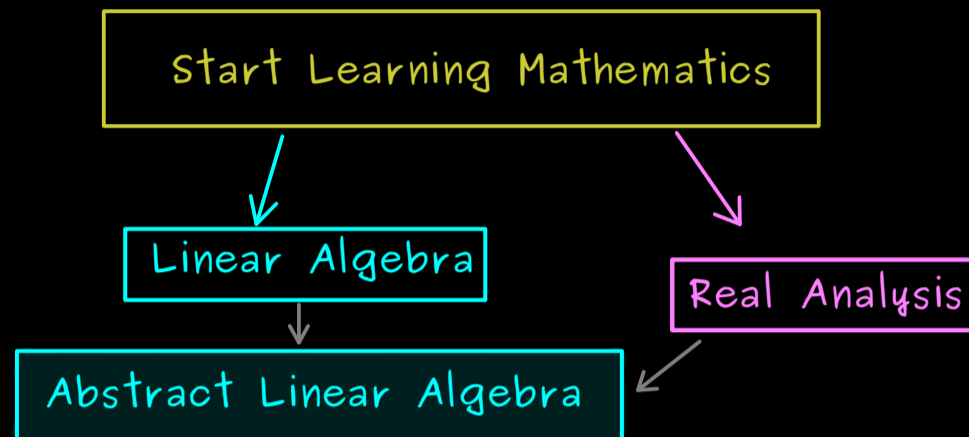
The Bright Side of Mathematics

The following pages cover the whole Abstract Linear Algebra course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: <https://tbsom.de/support>

Have fun learning mathematics!

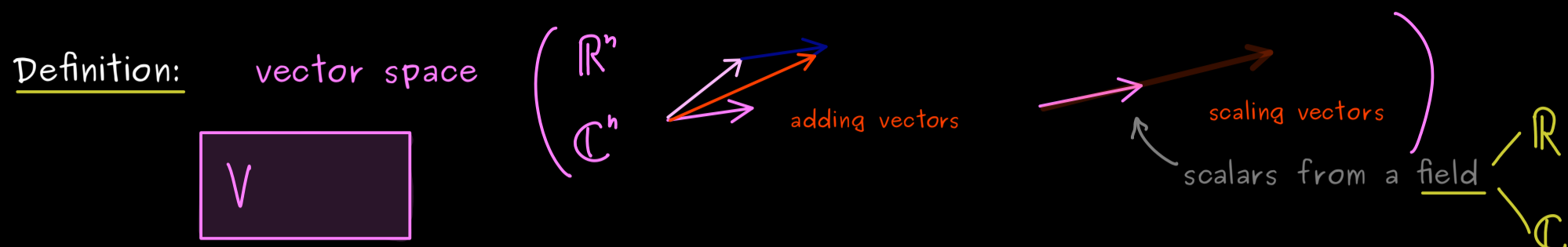
Abstract Linear Algebra – Part 1

Prerequisites:



Content:

- general vector spaces
- general linear maps
- change of basis
- general inner products
- eigenvalue theory for linear maps



Let \mathbb{F} be a field (often \mathbb{R} or \mathbb{C}).

A set $V \neq \emptyset$ together with two operations,

- vector addition $+$: $V \times V \rightarrow V$
- scalar multiplication \cdot : $\mathbb{F} \times V \rightarrow V$

where the following eight rules are satisfied, is called an \mathbb{F} -vector space.

(a) $(V, +)$ is an abelian group:

$$(1) \quad u + (v + w) = (u + v) + w \quad (\text{associativity of } +)$$

$$(2) \quad v + 0 = v \quad \text{with } 0 \in V \quad (\text{neutral element})$$

$$(3) \quad v + (-v) = 0 \quad \text{with } -v \in V \quad (\text{inverse elements})$$

$$(4) \quad v + w = w + v \quad (\text{commutativity of } +)$$

(b) scalar multiplication is compatible:

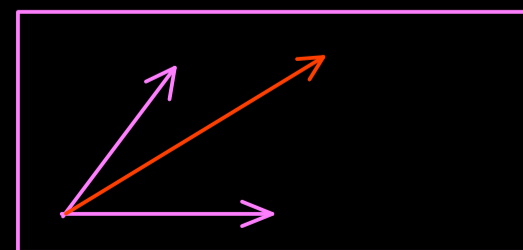
$$(5) \quad \lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$$

$$(6) \quad 1 \cdot v = v, \quad 1 \in \mathbb{F} \quad (\text{multiplicative unit from the field})$$

(c) distributive laws:

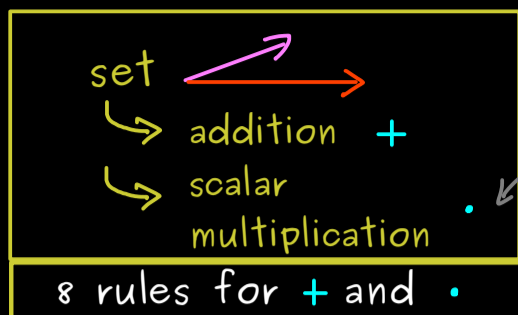
$$(7) \quad \lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$$

$$(8) \quad (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \quad \rightsquigarrow \text{abstract vector space}$$



Abstract Linear Algebra - Part 2

vector space:



field \mathbb{F}

(important cases: \mathbb{R}, \mathbb{C})

Examples: (a) The space of matrices $\mathbb{C}^{m \times n}$ with matrix addition and scaling:

complex vector space

(see: Linear Algebra - Part 11 and 58)

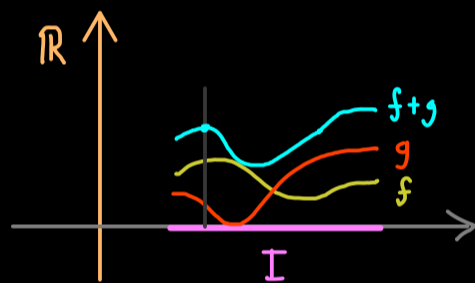
(b) Function space. Consider a set I and functions $f: I \rightarrow \mathbb{R}$.

Then $\mathcal{F}(I) := \{f: I \rightarrow \mathbb{R}\}$ defines a real vector space:

- vector addition $f + g$ defined by:

$$(f + g)(x) := f(x) + g(x)$$

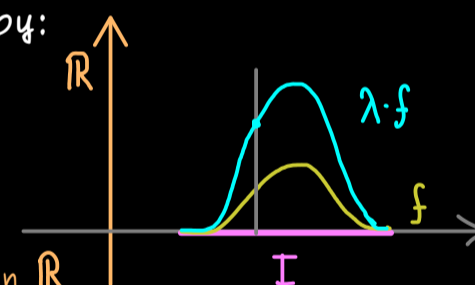
addition in \mathbb{R}



- scalar multiplication $\lambda \cdot f$ defined by:

$$(\lambda \cdot f)(x) := \lambda \cdot f(x)$$

multiplication in \mathbb{R}



↳ check 8 rules!

(c) space of polynomials: $\mathcal{P}(\mathbb{R}) := \{p: \mathbb{R} \rightarrow \mathbb{R} \text{ polynomial function}\}$

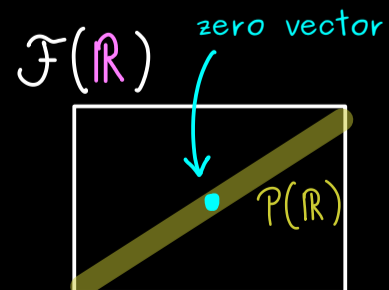
$$\hookrightarrow p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

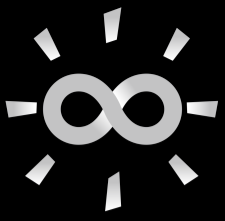
$p_1 + p_2, \lambda \cdot p$ defined as before

\Rightarrow real vector space

We see: $\mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$

linear subspace in $\mathcal{F}(\mathbb{R})$





Abstract Linear Algebra - Part 3

set + 8 rules

V \mathbb{F} -vector space

for example: space of functions

zero vector $0 \in V$

Question: $0 \cdot v = 0$ (zero in \mathbb{F}) ← zero vector, $(-1) \cdot v = -v$ for $v \in V$?

Proof:

$$0 \cdot v = (0 + 0) \cdot v \stackrel{(8)}{=} 0 \cdot v + 0 \cdot v$$

$$\stackrel{(3)}{\Rightarrow} 0 \cdot v + (-(0 \cdot v)) = 0 \cdot v + \underbrace{(0 \cdot v + (-(0 \cdot v)))}_{= 0} \stackrel{\text{associativity (1)}}{=} 0 \cdot v$$

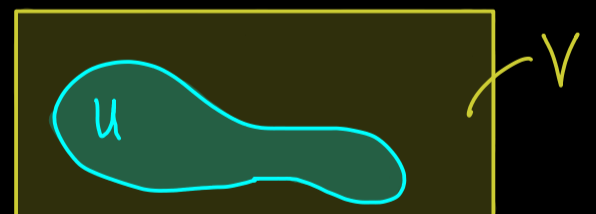
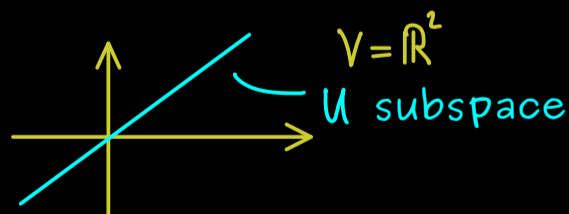
$$\stackrel{(3)}{\Rightarrow} 0 = 0 \cdot v$$

$$= (1 + (-1)) \cdot v \stackrel{(8)}{=} \underbrace{1 \cdot v}_{(6) \rightarrow v} + (-1) \cdot v$$

$$\stackrel{(3)}{\Rightarrow} -v + 0 = \underbrace{-v + v}_{= 0} + (-1) \cdot v \Rightarrow -v = (-1) \cdot v \quad \checkmark$$

Linear subspace:

• vector space inside another one



- $\mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$
 - zero function lies in $\mathcal{P}(\mathbb{R})$
 - adding two polynomials gives polynomial
 - scaling polynomial gives polynomial

Definition: V \mathbb{F} -vector space, $U \subseteq V$. If

(a) $0 \in U$,

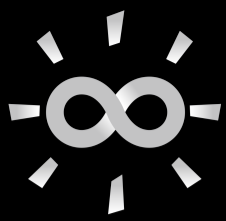
(b) $u, v \in U \Rightarrow u + v \in U$,

(c) $u \in U, \lambda \in \mathbb{F} \Rightarrow \lambda \cdot u \in U$,

then U is also an \mathbb{F} -vector space. We call it a linear subspace of V .

Example: $\mathcal{P}_2(\mathbb{R})$ polynomials with degree ≤ 2 ($x \mapsto 4x^2 + x$, $x \mapsto 8x + 1$)

$\Rightarrow \mathcal{P}_2(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$ subspace



Abstract Linear Algebra - Part 4

We know: $\mathcal{P}_k(\mathbb{R}) := \{ \text{polynomials with degree} \leq k \}$

$$\mathcal{P}_0(\mathbb{R}) \subseteq \mathcal{P}_1(\mathbb{R}) \subseteq \mathcal{P}_2(\mathbb{R}) \subseteq \dots \subseteq \mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$$

subspace subspace subspace subspace

Definition: V \mathbb{F} -vector space:

(a) For $v_1, \dots, v_k \in V$, $\alpha_1, \dots, \alpha_k \in \mathbb{F}$,

$$\sum_{j=1}^k \alpha_j v_j \quad \text{is called a linear combination.$$

(b) For subset $M \subseteq V$:

$$\text{Span}(M) := \{ \text{all possible linear combinations with vectors from } M \}$$

$$\text{Span}(\emptyset) := \{0\} \quad \leftarrow \text{subspace in } V$$

(c) A set $M \subseteq V$ is called a generating set of a subspace $U \subseteq V$ if

$$\text{Span}(M) = U$$

(d) A set $M \subseteq V$ is called a linearly independent if for all $k \in \mathbb{N}$ and $v_j \in V$:

$$0 = \sum_{j=1}^k \alpha_j v_j \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$$

(e) A set $M \subseteq V$ (or an ordered family $M = (v_1, \dots, v_k)$)

is called a basis of a subspace $U \subseteq V$ if M is generating and lin. independent.

(f) The number of elements in a basis of U is called the dimension of U

↑
cardinality of M

$$\dim(U) \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$$

↑
could be distinguished more

Example:

$$(1) \dim(\mathcal{P}_0(\mathbb{R})) = 1$$



space of constant functions/polynomials $\mathbb{R} \rightarrow \mathbb{R}$

$$\text{basis } M = (x \mapsto 1)$$

$$(2) \dim(\mathcal{P}_2(\mathbb{R})) = 3$$



polynomials of degree ≤ 2

$$\text{basis } M = (x \mapsto 1, x \mapsto x, x \mapsto x^2)$$

$$(3) \dim(\mathcal{F}(\mathbb{R})) = \infty$$

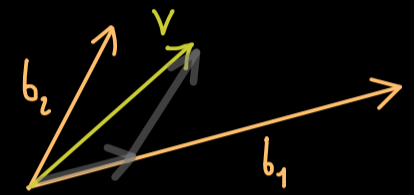
$$(4) \dim(\mathbb{C}^{2 \times 3}) = 6$$



Abstract Linear Algebra - Part 5

Coordinates with respect to a basis:

Assumptions: $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, V \mathbb{F} -vector space with $\dim(V) = n < \infty$,
 $\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V .



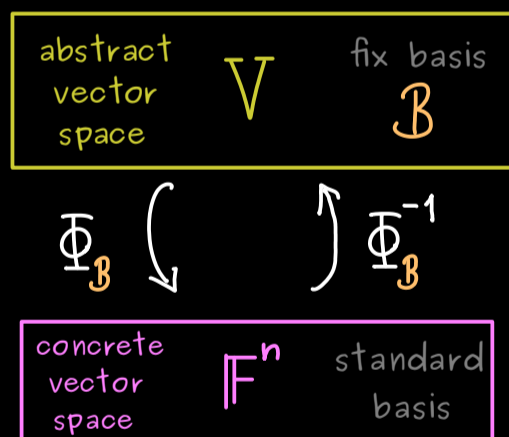
Then: each vector $v \in V$ can be uniquely

written as: $v = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_n b_n$ with $\alpha_j \in \mathbb{F}$

Definition: α_j are called the coordinates of v with respect to \mathcal{B} .

Remember: $v = \sum_{j=1}^n \alpha_j b_j \iff \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n$
coordinate vector

Picture:



Define: $\Phi_{\mathcal{B}}(\alpha_1 b_1 + \dots + \alpha_n b_n) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

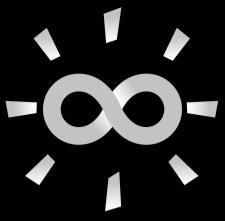
$\Phi_{\mathcal{B}} : V \longrightarrow \mathbb{F}^n$ is a linear map:

$$\Phi_{\mathcal{B}}(v+w) = \Phi_{\mathcal{B}}(v) + \Phi_{\mathcal{B}}(w)$$

$$\Phi_{\mathcal{B}}(\lambda \cdot v) = \lambda \cdot \Phi_{\mathcal{B}}(v)$$

$\Phi_{\mathcal{B}}$ is called basis isomorphism

$\hookrightarrow \Phi_{\mathcal{B}}(b_j) = e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ canonical unit vector



Abstract Linear Algebra - Part 6

subset of $\mathcal{F}(\mathbb{R})$ given by:

$$\cos: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \cos$$

$$\sin: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \sin$$

$$\exp: \mathbb{R} \rightarrow \mathbb{R} \rightsquigarrow \text{graph of } \exp$$

$$U := \text{Span}(\cos, \sin, \exp)$$

Question: Is (\cos, \sin, \exp) a basis of U ?

generating ✓

linearly independent?

We have to check: $\alpha_1 \cdot \cos + \alpha_2 \cdot \sin + \alpha_3 \cdot \exp = 0 \Rightarrow \alpha_j = 0$ for all j

means:

zero vector in $\mathcal{F}(\mathbb{R})$

$$\alpha_1 \cdot \cos(x) + \alpha_2 \cdot \sin(x) + \alpha_3 \cdot \exp(x) = 0(x)$$

$$\begin{aligned} \hookrightarrow 0: \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

for all $x \in \mathbb{R}$

$$\Rightarrow \begin{cases} \alpha_1 \cdot \cos(0) + \alpha_2 \cdot \sin(0) + \alpha_3 \cdot \exp(0) = 0 \\ \alpha_1 \cdot \cos\left(\frac{\pi}{2}\right) + \alpha_2 \cdot \sin\left(\frac{\pi}{2}\right) + \alpha_3 \cdot \exp\left(\frac{\pi}{2}\right) = 0 \\ \alpha_1 \cdot \cos(-2\pi \cdot 500) + \alpha_2 \cdot \sin(-2\pi \cdot 500) + \alpha_3 \cdot \exp(-2\pi \cdot 500) = 0 \end{cases}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right. \text{system of linear equations}$$

since $\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} = e^{-1000\pi} + 0 + 0 - 1 - 0 - 0 < 0$,

the system of linear equations is uniquely solvable.

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \Rightarrow \mathcal{B} = (\cos, \sin, \exp) \text{ basis of } \mathcal{U}$$

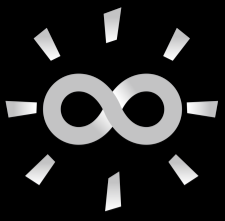
Basis isomorphism: $\Phi_{\mathcal{B}} : \mathcal{U} \rightarrow \mathbb{R}^3$,

defined by $\Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\sin) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\exp) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

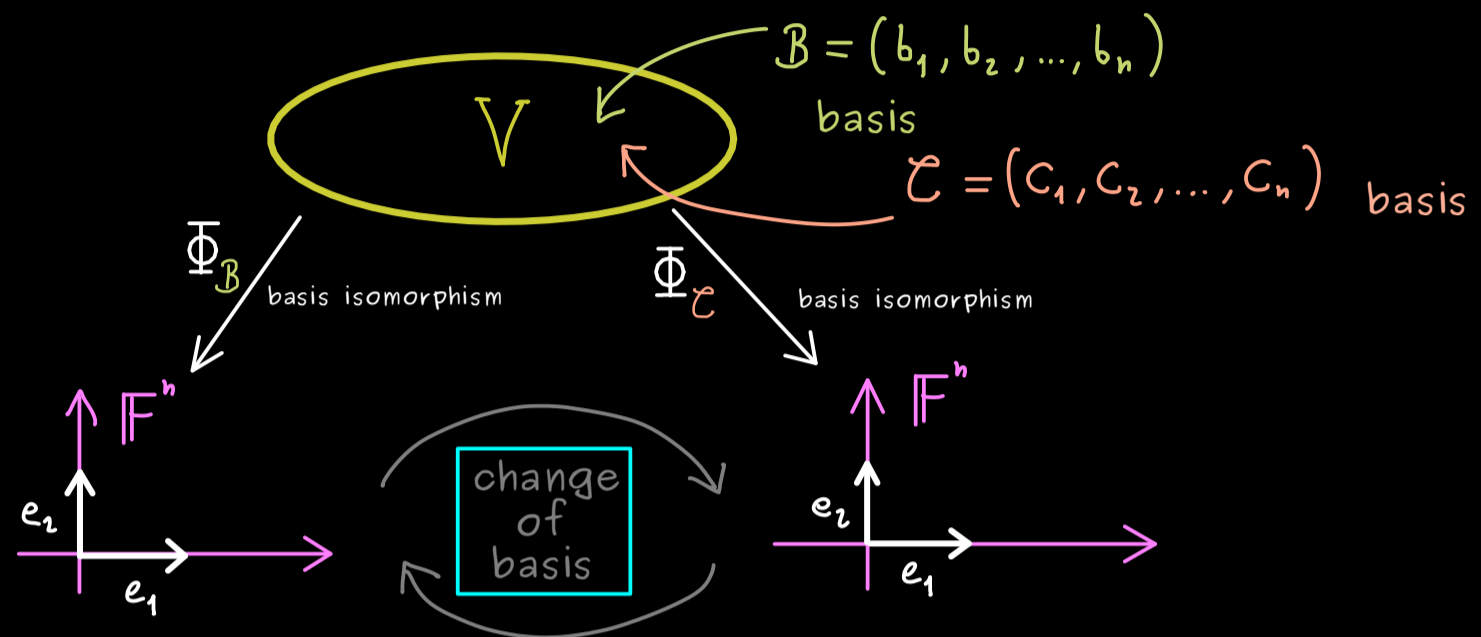
What about $v: \mathbb{R} \rightarrow \mathbb{R}$, $v(x) = 7 \cos(x) + 2 \exp(x)$

$$\Phi_{\mathcal{B}}(v) = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

\mathcal{U} is completely represented by \mathbb{R}^3



Abstract Linear Algebra - Part 7



Recall: $\Phi_B : V \rightarrow \mathbb{F}^n$ given by $\Phi_B(b_j) = e_j$ for all j

$\Phi_B^{-1} : \mathbb{F}^n \rightarrow V$ given by $\Phi_B^{-1}(e_j) = b_j$ for all j

For each $v \in V$: $v = \Phi_B^{-1}\left(\begin{pmatrix} \text{coordinate} \\ \text{vector} \end{pmatrix}\right)$

Example: $\mathcal{P}_2(\mathbb{R})$ with basis $B = (m_0, m_1, m_2)$ where $m_0(x) = 1$, $m_1(x) = x$, $m_2(x) = x^2$

For $p \in \mathcal{P}_2(\mathbb{R})$ given $p(x) = 3x^2 + 8x - 2$

$$p = (-2) \cdot m_0 + 8 \cdot m_1 + 3 \cdot m_2 = \Phi_B^{-1}\left(\begin{pmatrix} -2 \\ 8 \\ 3 \end{pmatrix}\right) \quad \text{coordinate vector}$$

Another basis: $C = (c_1, c_2, c_3)$ with $c_1 = m_0$, $c_2 = m_1$, c_3 polynomial

$\hookrightarrow c_3(x) = 3x^2 + 8x$

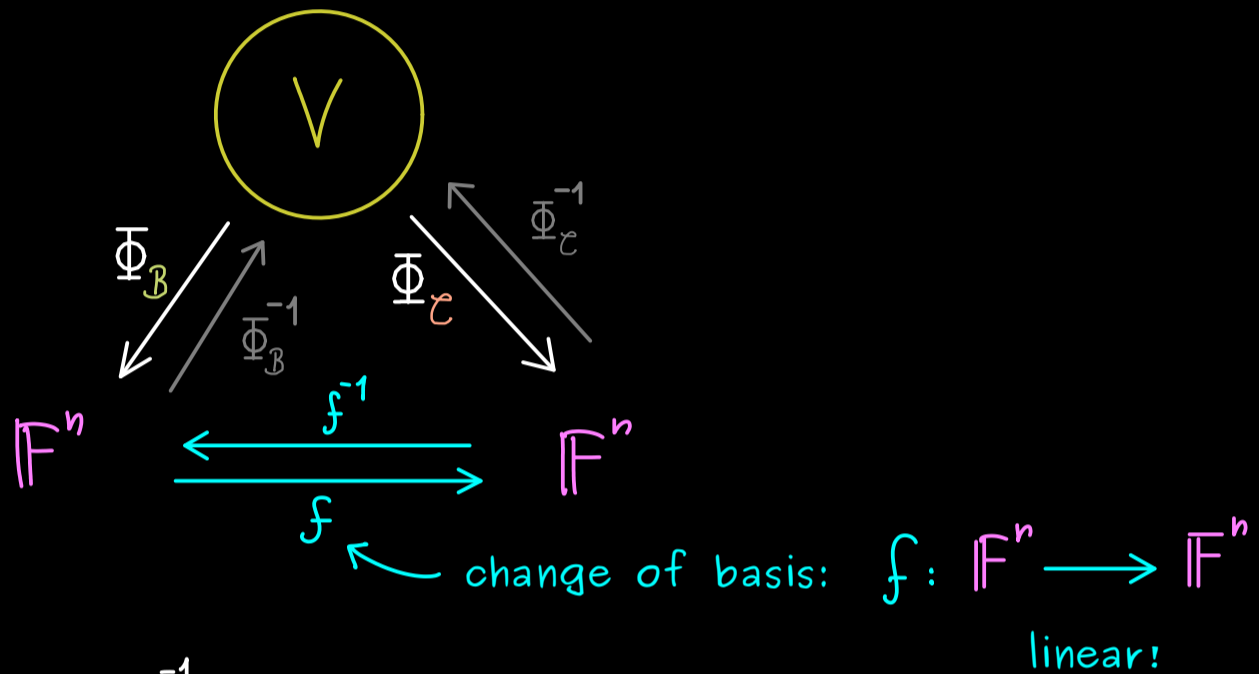
$$p = \Phi_C^{-1}\left(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right) \quad \text{coordinate vector}$$

Old vs. new coordinates: $\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis, $\mathcal{C} = (c_1, c_2, \dots, c_n)$ basis

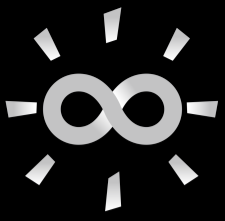
$$\Phi_{\mathcal{B}}(v) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \longleftrightarrow \Phi_{\mathcal{C}}(v) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$v = \beta_1 b_1 + \dots + \beta_n b_n$$

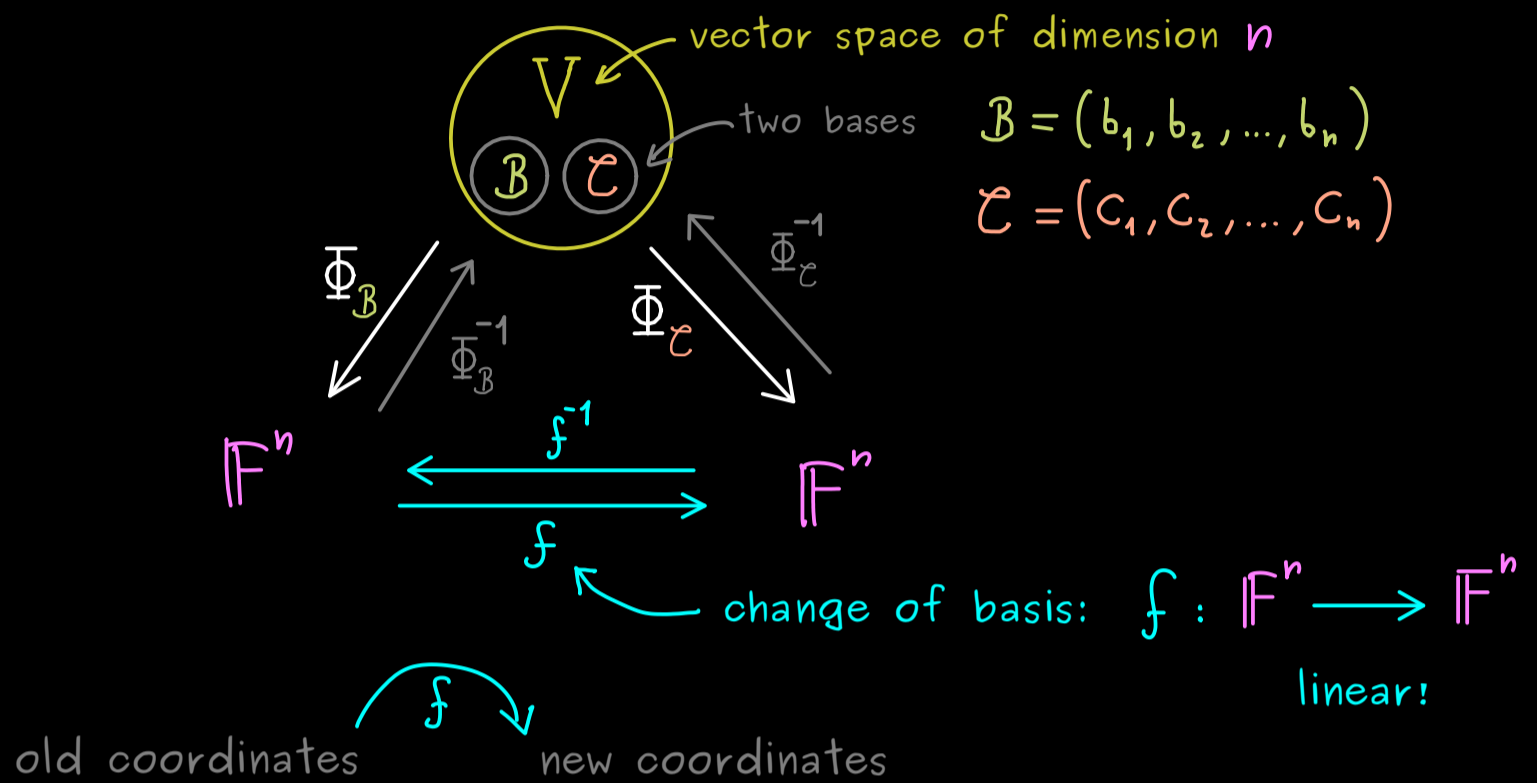
$$v = \gamma_1 c_1 + \dots + \gamma_n c_n$$



We get: $f(x) = \Phi_{\mathcal{C}} \circ \Phi_{\mathcal{B}}^{-1}(x)$



Abstract Linear Algebra - Part 8

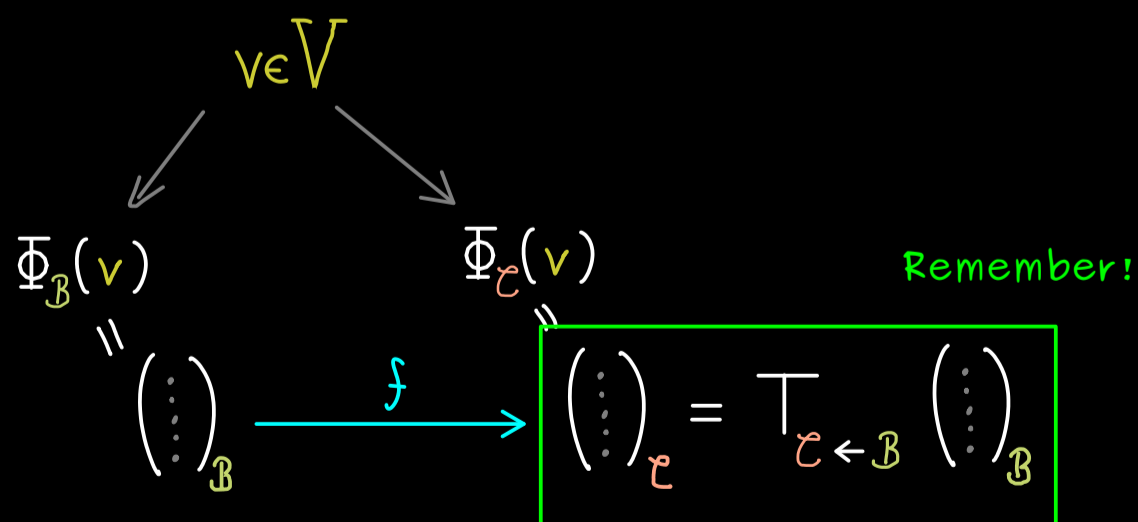


What happens if we put $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ into f ? $\leadsto f(e_1) = \Phi_{\mathcal{C}} \left(\underbrace{\Phi_{\mathcal{B}}^{-1}(e_1)}_{b_1} \right) = \Phi_{\mathcal{C}}(b_1)$

We can see f as a matrix $f(x) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} | \\ | \\ | \\ | \end{pmatrix} x$

$$T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(b_1) & \Phi_{\mathcal{C}}(b_2) & \dots & \Phi_{\mathcal{C}}(b_n) \\ | & | & & | \end{pmatrix}$$

transformation matrix
transition matrix
change-of-basis matrix
from \mathcal{B} to \mathcal{C}



Fact: $\left(T_{\mathcal{C} \leftarrow \mathcal{B}} \right)^{-1} = T_{\mathcal{B} \leftarrow \mathcal{C}}$

Example: $V = \mathcal{P}_2(\mathbb{R})$ polynomials of degree ≤ 2

$$m_0: X \mapsto 1$$

$$m_1: X \mapsto X$$

$$m_2: X \mapsto X^2$$

$$\mathcal{B} = (\underbrace{m_2}_{b_1}, \underbrace{m_1}_{b_2}, \underbrace{m_0}_{b_3})$$

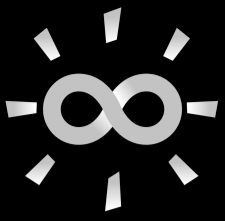
$$\mathcal{C} = (\underbrace{m_2 - \frac{1}{2}m_1}_{c_1}, \underbrace{m_2 + \frac{1}{2}m_1}_{c_2}, \underbrace{m_0}_{c_3})$$

$T_{\mathcal{C} \leftarrow \mathcal{B}}$ \rightsquigarrow how to write b_j with a linear combination of \mathcal{C}

$T_{\mathcal{B} \leftarrow \mathcal{C}}$ \rightsquigarrow how to write c_j with a linear combination of \mathcal{B}

\hookrightarrow column vectors $\Phi_{\mathcal{B}}(c_1) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$T_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{calculate inverse!}} T_{\mathcal{C} \leftarrow \mathcal{B}}$$



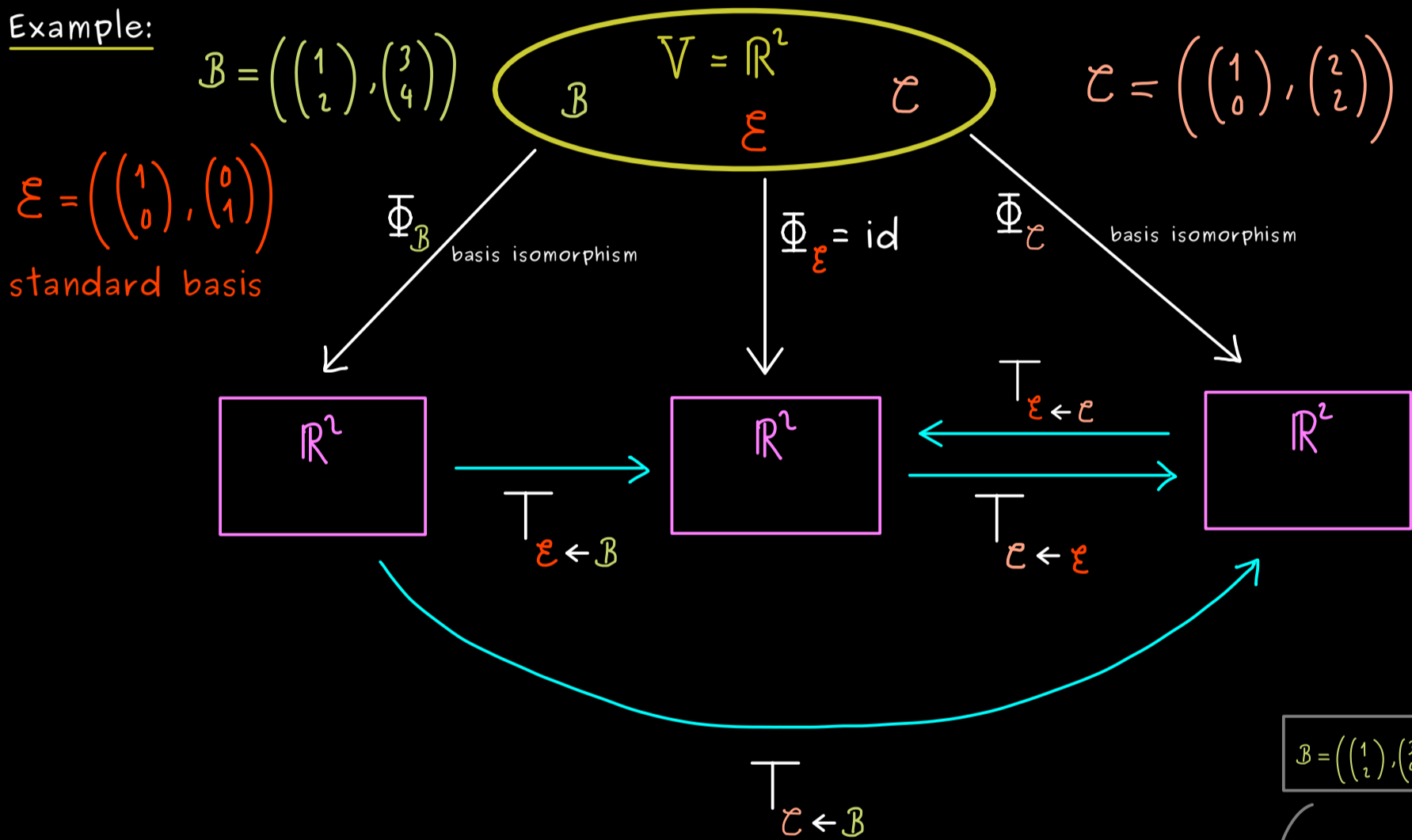
Abstract Linear Algebra - Part 9

$$\underbrace{T_{\mathcal{B} \leftarrow \mathcal{C}}}_{\text{change-of-basis matrix}} \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}_{\mathcal{C}} = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}_{\mathcal{B}}$$

$\Phi_{\mathcal{C}}(v)$ $\Phi_{\mathcal{B}}(v)$

V vector space of dimension n
 \Downarrow
 $T_{\mathcal{B} \leftarrow \mathcal{C}}$ $(n \times n)$ -matrix
 invertible

Example:



We already know:

$$T_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(b_1) & \Phi_{\mathcal{E}}(b_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$T_{\mathcal{E} \leftarrow \mathcal{C}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(c_1) & \Phi_{\mathcal{E}}(c_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ c_1 & c_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$

We can calculate:

$$\begin{aligned} T_{\mathcal{C} \leftarrow \mathcal{B}} &= T_{\mathcal{C} \leftarrow \mathcal{E}} T_{\mathcal{E} \leftarrow \mathcal{B}} \\ &= \underbrace{\left(T_{\mathcal{E} \leftarrow \mathcal{C}} \right)^{-1}} T_{\mathcal{E} \leftarrow \mathcal{B}} \end{aligned}$$

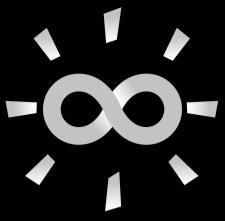
calculate product immediately!

$$\begin{aligned} T_{\mathcal{E} \leftarrow \mathcal{C}} X &= T_{\mathcal{E} \leftarrow \mathcal{B}} \\ \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \text{solve } \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array} \right) \xrightarrow{\text{II} \cdot \frac{1}{2}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array} \right)$$

$$\xrightarrow{\text{I} - 2\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array} \right)$$

$$X = T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$



Abstract Linear Algebra - Part 10

Always: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

$$\bar{\alpha} := \begin{cases} \alpha, & \mathbb{F} = \mathbb{R} \\ \bar{\alpha}, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } \alpha \in \mathbb{F}$$
$$A^* := \begin{cases} A^T, & \mathbb{F} = \mathbb{R} \\ A^*, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } A \in \mathbb{F}^{m \times n}$$

Definition: $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$

is called an inner product on the \mathbb{F} -vector space V if:

(1) $\langle x, x \rangle \geq 0$ for all $x \in V$ (positive definite)

and $\langle x, x \rangle = 0 \implies x = 0$ (zero vector)

(2) $\langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle$ for all $x, \tilde{x}, y \in V$

$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in V$

(linear in the second argument)

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$ (conjugate symmetric)

Example: (a) For $u, v \in \mathbb{F}^n$, define:

$$\langle u, v \rangle_{\text{standard}} := \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n = u^* v$$

(b) For $u, v \in \mathbb{F}^2$, define:

$$\langle u, v \rangle = \bar{u}_1 v_2 + \bar{u}_2 v_1 \rightsquigarrow (2) \text{ and } (3) \text{ satisfied}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -1 - 1 = -2 < 0 \rightsquigarrow (1) \text{ not satisfied}$$

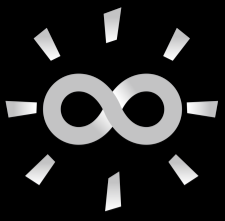
not an inner product!

(c) $\mathcal{P}([0,1], \mathbb{F})$ polynomial space, $p(x) = ix$ is in $\mathcal{P}([0,1], \mathbb{F})$

Define: $\langle f, g \rangle = \int_0^1 \overline{f(x)} g(x) dx$

Example: $\langle p, p \rangle = \int_0^1 \overline{ix} \cdot ix dx = \int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$

$$\left(\sum_{i=1}^n \bar{u}_i v_i \rightsquigarrow \int_0^1 \overline{f \cdot g} \right)$$



Abstract Linear Algebra - Part 11

Example: In \mathbb{F}^2 :

$$\begin{aligned} \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle &= \bar{u}_1 v_1 + \bar{u}_1 v_2 + \bar{u}_2 v_1 + 4 \bar{u}_2 v_2 \\ &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}}_A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_{\text{standard}} \end{aligned}$$

→ check 3 rules of inner product

$$\hookrightarrow \langle x, x \rangle = \langle x, Ax \rangle_{\text{standard}} > 0 \quad \text{for } x \neq 0$$

Definition: $A \in \mathbb{F}^{n \times n}$ is called a positive definite matrix if:

- $A^* = A$ (selfadjoint/symmetric)
- $\langle x, Ax \rangle_{\text{standard}} > 0$ for all $x \in \mathbb{F}^n \setminus \{0\}$

Fact: If $A \in \mathbb{F}^{n \times n}$ is a positive definite matrix, then

$$\langle y, x \rangle := \langle y, Ax \rangle_{\text{standard}} \quad \text{defines an inner product in } \mathbb{F}^n.$$

Example: $\langle x, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} x \rangle_{\text{standard}} = \bar{x}_1 x_1 + \bar{x}_1 x_2 + \bar{x}_2 x_1 + 4 \bar{x}_2 x_2$
 $= |x_1 + x_2|^2 + 3 \cdot |x_2|^2 \geq 0$

If $|x_1 + x_2|^2 + 3 \cdot |x_2|^2 = 0 \Rightarrow |x_1 + x_2|^2 = 0$ and $|x_2|^2 = 0$
 $\Rightarrow x_2 = 0$
 $\Rightarrow x_1 = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ positive definite

Proposition: For a selfadjoint matrix $A \in \mathbb{F}^{n \times n}$, the following claims are equivalent:

- (a) A positive definite
- (b) All eigenvalues of A are positive (> 0)
- (c) After Gaussian elimination (without scaling and exchanging rows) only with row operations $Z_i + \lambda_j$, (see part 37 of Linear Algebra) all pivots in the row echelon form are positive.

- (d) The determinants of the so-called leading principal minors of A are positive.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$H_1 = (a_{11})$, $H_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,
 $H_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \dots, H_n = A$

$$\det(H_1) > 0, \det(H_2) > 0, \dots, \det(H_n) > 0$$

(Sylvester's criterion)

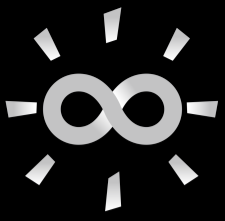
Example:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

(d) $\det(1) = 1 > 0$

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 4 - 1 = 3 > 0$$

(c) Gaussian elimination: $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{\mathbb{I} - 1\mathbb{I}} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$



Abstract Linear Algebra - Part 12

Recall: inner product on the \mathbb{F} -vector space V :

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F} \quad \text{three properties!}$$

For $V = \mathbb{F}^n$: $\langle \gamma, x \rangle = \langle \gamma, Ax \rangle_{\text{standard}}$
positive definite matrix

We use inner products for:

- measuring angles \leftarrow Cauchy Schwarz inequality

- measuring lengths: $\|x\| := \sqrt{\langle x, x \rangle}$
norm of x

Cauchy-Schwarz inequality: $\langle \cdot, \cdot \rangle$ inner product on the \mathbb{F} -vector space V .

Then: $|\langle \gamma, x \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in V$

and $|\langle \gamma, x \rangle| = \|x\| \cdot \|y\| \iff x, y$ lin. dependent

Proof: (1) For $x=0$: $\langle \gamma, \underbrace{x}_{0 \cdot v} \rangle = 0 \cdot \langle \gamma, v \rangle = 0$ and $\|x\| \cdot \|y\| = 0$

(2) For $x \neq 0$: Show: $|\langle y, \underbrace{\frac{x}{\|x\|}}_{\hat{x}} \rangle| \leq \|y\|$, $\|\hat{x}\| = 1$

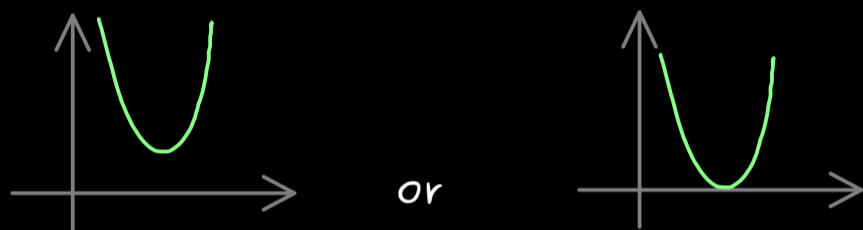
For any $\lambda \in \mathbb{R}$: $0 \leq \langle y - \lambda \hat{x}, y - \lambda \hat{x} \rangle$

$$= \langle y, y \rangle - \underbrace{\lambda \langle \hat{x}, y \rangle}_{\bar{a}} - \underbrace{\lambda \langle y, \hat{x} \rangle}_{a} + \lambda^2 \langle \hat{x}, \hat{x} \rangle$$

$$= \lambda^2 + \lambda \cdot \underbrace{(-2 \cdot \text{Re}(\langle y, \hat{x} \rangle))}_{p} + \underbrace{\|y\|^2}_{q}$$

2. $\langle y, \hat{x} \rangle$

quadratic polynomial has zeros: $\lambda_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$

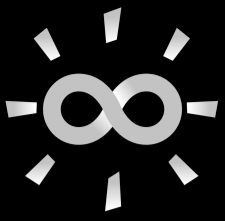


$$\Rightarrow \left(\frac{p}{2}\right)^2 - q \leq 0 \Rightarrow \text{Re}(\langle y, \hat{x} \rangle)^2 \leq \|y\|^2$$

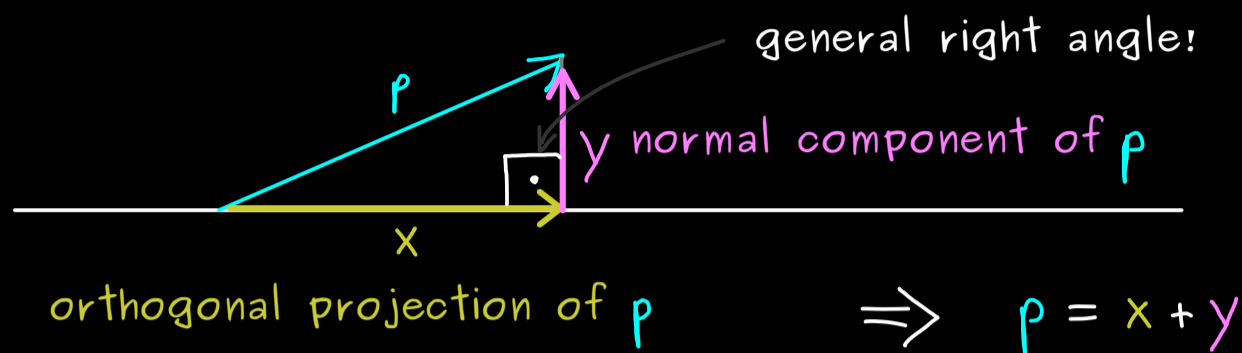
$$\Rightarrow |\text{Re}(\langle y, \hat{x} \rangle)| \leq \|y\| \quad \rightsquigarrow \text{Cauchy-Schwarz } \mathbb{F} = \mathbb{R}$$

For $\mathbb{F} = \mathbb{C}$: $\underbrace{e^{i\psi}}_c \langle y, \hat{x} \rangle = |\langle y, \hat{x} \rangle|$

$$|\text{Re}(c \langle y, \hat{x} \rangle)| = |\text{Re}(\langle y, \underbrace{c \hat{x}}_{\hat{x}} \rangle)| \leq \|y\|$$



Abstract Linear Algebra - Part 13



Definition: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

We say $x, y \in V$ are orthogonal, written as $x \perp y$, if $\langle x, y \rangle = 0$.

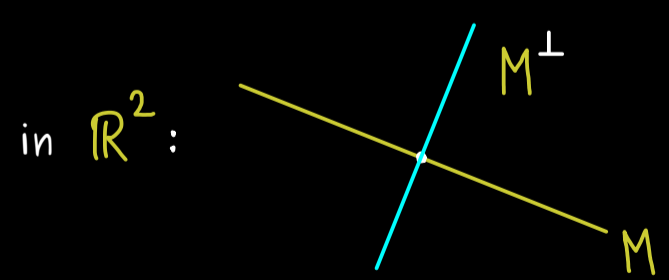
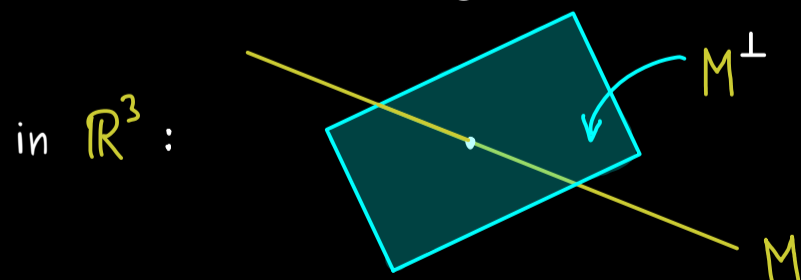
Example: $\mathcal{P}([-1, 1], \mathbb{F})$ polynomial space, $\langle f, g \rangle = \int_{-1}^1 \overline{f(x)} g(x) dx$

$p_1 : x \mapsto x$
 $p_2 : x \mapsto x^2 \Rightarrow \langle p_1, p_2 \rangle = \int_{-1}^1 x^3 dx = 0 \Rightarrow p_1 \perp p_2$

Definition: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

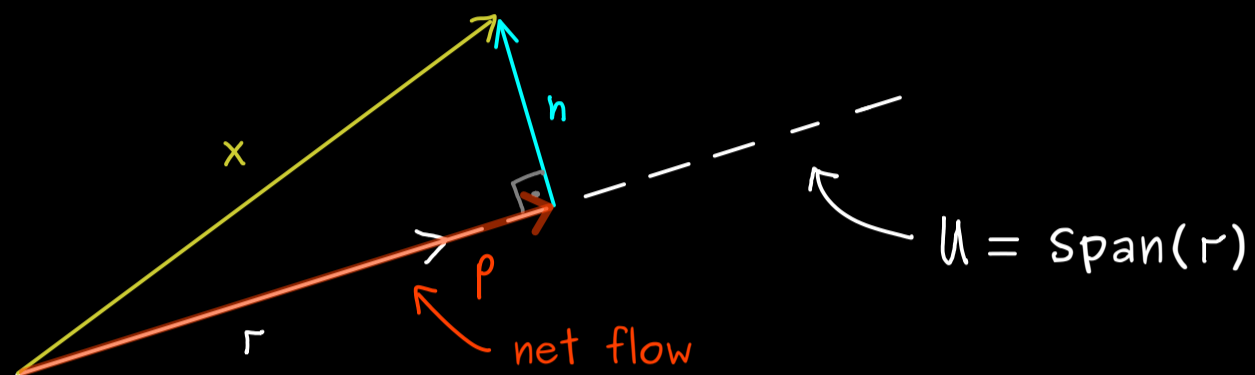
For $M \subseteq V$, $M \neq \emptyset$, we define the orthogonal complement:

$$M^\perp := \left\{ x \in V \mid \langle x, m \rangle = 0 \text{ for all } m \in M \right\}$$





Abstract Linear Algebra - Part 14



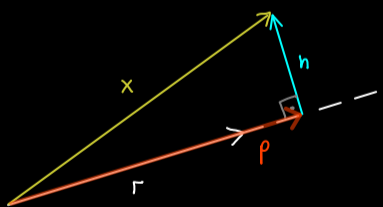
Definition: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$.

Let $U \subseteq V$ be a subspace with $U = \text{Span}(r)$, $r \neq 0$.

For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \perp r$, we call:

p orthogonal projection of x onto U
 n normal component of x with respect to U

Let's show the uniqueness: Assume $x = p + n$, $x = \tilde{p} + \tilde{n}$
 $\in U$ $\in U^\perp$ $\in U$ $\in U^\perp$



$$\Rightarrow p + n = \tilde{p} + \tilde{n} \Rightarrow \underbrace{p - \tilde{p}}_{\in U} = \underbrace{\tilde{n} - n}_{\in U^\perp}$$

$$\Rightarrow 0 = \langle p - \tilde{p}, \tilde{n} - n \rangle = \begin{cases} \langle p - \tilde{p}, p - \tilde{p} \rangle \\ \langle \tilde{n} - n, \tilde{n} - n \rangle \end{cases}$$

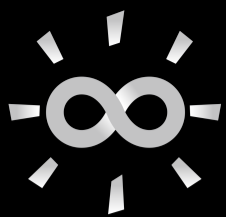
inner product is positive definite

$$\Rightarrow p - \tilde{p} = 0 = \tilde{n} - n \Rightarrow p = \tilde{p} \text{ and } n = \tilde{n}$$

Existence: $p \in U = \text{Span}(r) \implies p = \lambda \cdot r$ for $\lambda \in \mathbb{F}$

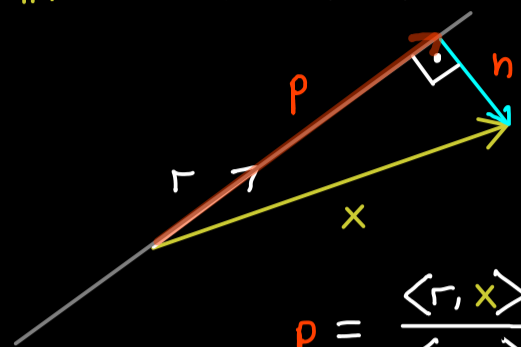
$$\langle r, x \rangle = \langle r, \lambda \cdot r + n \rangle = \lambda \langle r, r \rangle + \underbrace{\langle r, n \rangle}_{=0}$$

$$\implies \lambda = \frac{\langle r, x \rangle}{\langle r, r \rangle} \implies p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r, \quad n = x - p \quad \checkmark$$



Abstract Linear Algebra - Part 15

Example: \mathbb{R}^2 with standard inner product.



$$r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{r} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(unit vector)

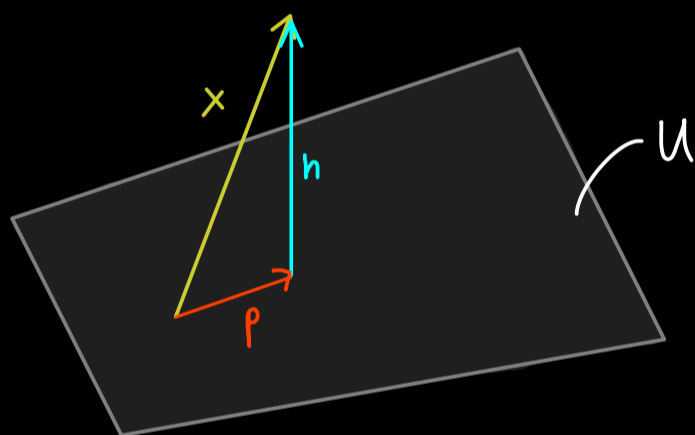
$$x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r = \langle \hat{r}, x \rangle \cdot \hat{r} \quad \left(\begin{array}{l} \text{orthogonal projection:} \\ \hat{r} \langle \hat{r}, \cdot \rangle \end{array} \right)$$

$$\Rightarrow p = \frac{1}{\sqrt{2}} (5+4) \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix}$$

$$\Rightarrow n = x - p = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

Generalization:



$$x = p + n \quad \begin{matrix} p \in U \\ n \in U^\perp \end{matrix}$$

Important fact: $U \cap U^\perp = \{0\}$ for every subspace $U \subseteq V$

Proposition: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$.

Let $U \subseteq V$ be a k -dimensional subspace, $\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U .

Then for $y \in V$: $y \perp U$ for all $u \in U$

\Leftrightarrow

$$y \perp b_j \text{ for all } j \in \{1, 2, \dots, k\}$$

Proof: $(\Rightarrow) \checkmark$ (\Leftarrow) We assume: $\langle y, b_j \rangle = 0$ for all $j \in \{1, 2, \dots, k\}$

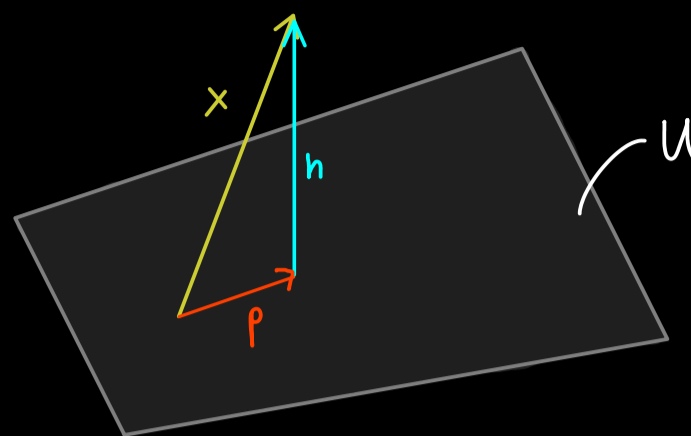
$$\Rightarrow \sum_{j=1}^k \lambda_j \langle y, b_j \rangle = 0$$

$$\Rightarrow \left\langle y, \sum_{j=1}^k \lambda_j b_j \right\rangle = 0 \quad \begin{matrix} \mathcal{B} \text{ basis} \\ \Rightarrow y \perp u \\ \text{for all } u \in U \end{matrix}$$

Orthogonal projection onto a subspace:

V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$,

$U \subseteq V$ k -dimensional subspace.

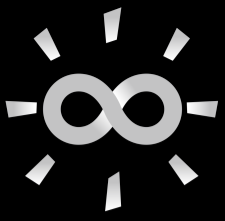


For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \in U^\perp$,

we call:

p orthogonal projection of x onto U

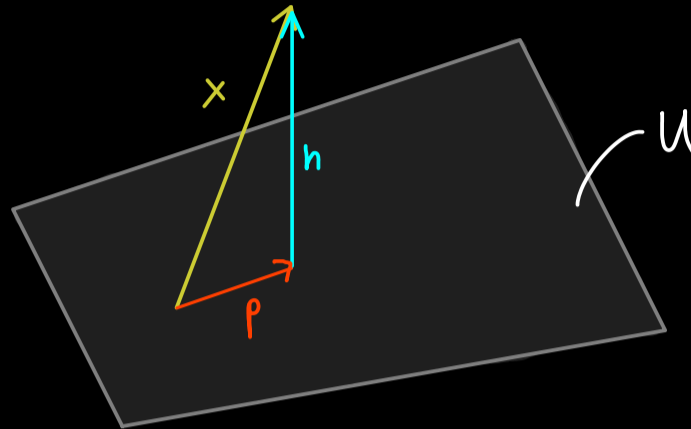
n normal component of x with respect to U



Abstract Linear Algebra - Part 16

Orthogonal projection:

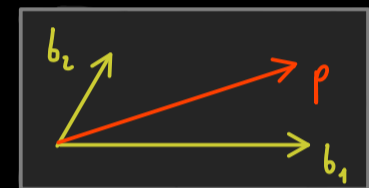
V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$,
 $U \subseteq V$ k -dimensional subspace.



$$x = \underset{\in U}{p} + \underset{\in U^\perp}{n}$$

Assume we have a basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$ of U .

$$p = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k \quad \text{for some } \lambda_1, \dots, \lambda_k \in \mathbb{F}$$



$$\begin{aligned} \text{For each basis vector } b_j : \langle b_j, x \rangle &= \langle b_j, p \rangle + \underbrace{\langle b_j, n \rangle}_{=0} \\ &= \langle b_j, \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k \rangle \\ &= \sum_{i=1}^k \lambda_i \langle b_j, b_i \rangle \end{aligned}$$

Let's rewrite these k linear equations:

$$\underbrace{\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \dots & \langle b_k, b_k \rangle \end{pmatrix}}_{\text{Gramian matrix } G(\mathcal{B})} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

\leadsto solution gives us the orthogonal projection

Do we have a unique solution? $G(\mathcal{B})$ invertible $\Leftrightarrow \text{Ker}(G(\mathcal{B})) = \{0\}$

Let's prove $\text{Ker}(G(\mathcal{B})) = \{0\}$: Choose $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \text{Ker}(G(\mathcal{B}))$

$$G(\mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \text{for all } j: \beta_1 \langle b_j, b_1 \rangle + \beta_2 \langle b_j, b_2 \rangle + \dots + \beta_k \langle b_j, b_k \rangle = 0$$

linearity

$$\Rightarrow \text{for all } j: \langle b_j, \underbrace{\sum_{i=1}^k \beta_i b_i}_{\gamma \in U} \rangle = 0$$

Proposition
part 15

$$\Rightarrow \gamma \in U^\perp$$

$$U \cap U^\perp = \{0\} \Rightarrow \gamma = 0 \Rightarrow \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example: \mathbb{R}^3 with standard inner product, $U = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

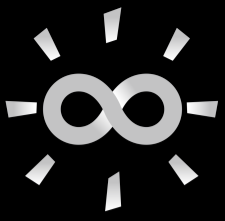
$$x = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow G(\mathcal{B}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \dots & \langle b_k, b_k \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

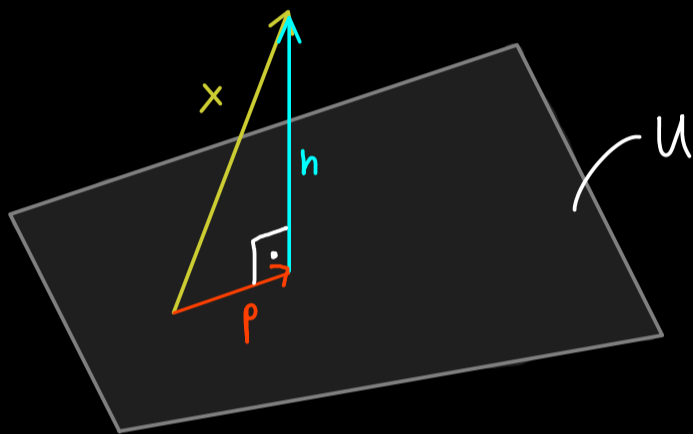
$$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 1 \end{array} \right) \xrightarrow{2 \cdot \text{II}} \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right)$$

$$\Rightarrow p = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right) \rightsquigarrow \begin{matrix} \lambda_2 = -1 \\ \lambda_1 = 2 \end{matrix}$$



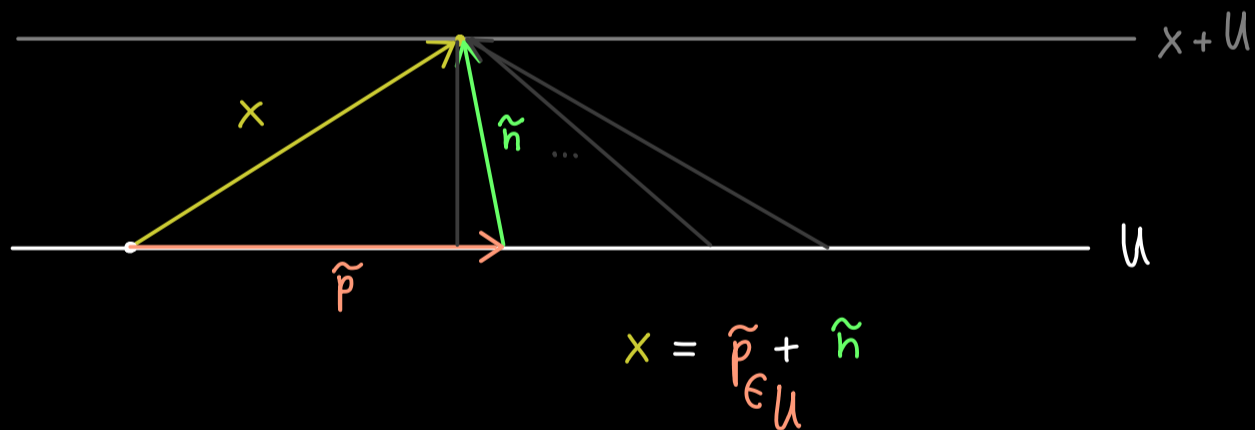
Abstract Linear Algebra - Part 17

V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.



$$\begin{aligned} x &= p + n \in U^\perp \\ &= x|_U + x|_{U^\perp} \end{aligned}$$

Simplified picture: What is the distance between U and $x + U$?



Approximation formula:

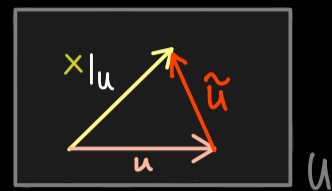
V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

For $x \in V$: $\text{dist}(x, U) := \inf \{ \|x - u\| \mid u \in U \} = \|x - \underbrace{x|_U}_{\text{orthogonal projection}}\|$

orthogonal projection

Recall: $\|x\| := \sqrt{\langle x, x \rangle}$
norm of x

Proof: For all $u \in U$: $\|x - u\|^2 = \left\| \underbrace{(x - x|_U)}_n + \underbrace{(x|_U - u)}_{=: \tilde{u} \in U} \right\|^2$



normal component of x with respect to U

$$= \langle n + \tilde{u}, n + \tilde{u} \rangle$$

$$= \langle n, n \rangle + \underbrace{\langle n, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, n \rangle}_{=0} + \langle \tilde{u}, \tilde{u} \rangle$$

$n \in U^\perp$

$$= \|n\|^2 + \underbrace{\|\tilde{u}\|^2}_{\geq 0} \geq \|n\|^2$$

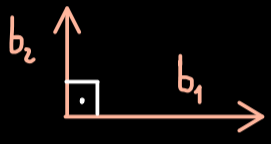
$$\Rightarrow \inf \{ \|x - u\| \mid u \in U \} \geq \|n\|$$

We have equality $\Leftrightarrow \tilde{u} = 0 \Leftrightarrow u = x|_U \quad \square$



Abstract Linear Algebra - Part 18

Assumption: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

Idea: Choose a nice basis (b_1, b_2, \dots, b_k) of U : 
 $\langle b_1, b_2 \rangle = 0$
 $\langle b_1, b_1 \rangle = \|b_1\|^2 = 1$, $\langle b_2, b_2 \rangle = 1$

Notation: $\langle b_i, b_j \rangle = \delta_{ij} := \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$
Kronecker delta

Orthogonal projection: For $x \in V$: $x = x|_U + x|_{U^\perp}$ can be calculated:
 $\in U$ $\in U^\perp$

$\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U

$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix} \rightsquigarrow$ solving LES gives $x|_U$
Gramian matrix

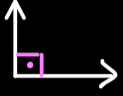
$$G(\mathcal{B}) = \begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \dots & \langle b_k, b_k \rangle \end{pmatrix} \stackrel{\text{nice basis}}{\downarrow} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

identity matrix

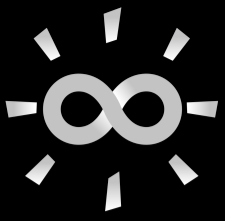
$$\Rightarrow x|_U = \sum_{j=1}^k b_j \langle b_j, x \rangle$$

Definition: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

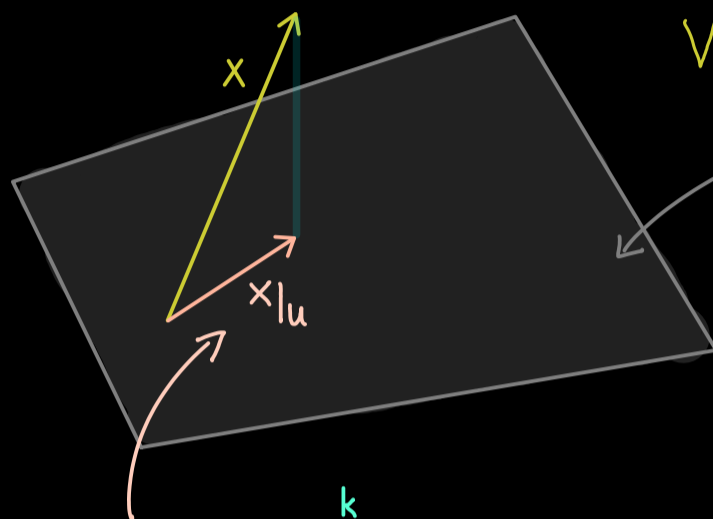
A family (b_1, b_2, \dots, b_m) (with $b_j \in U$) is called:

- orthogonal system (OS) if $\langle b_i, b_j \rangle = 0$ for all $i \neq j$ 
- orthonormal system (ONS) if $\langle b_i, b_j \rangle = \delta_{ij}$
- orthogonal basis (OB) if it's an OS and a basis of U
- orthonormal basis (ONB) if it's an ONS and a basis of U

Example: \mathbb{R}^3 with standard inner product, $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ ONB of \mathbb{R}^3 .



Abstract Linear Algebra - Part 19



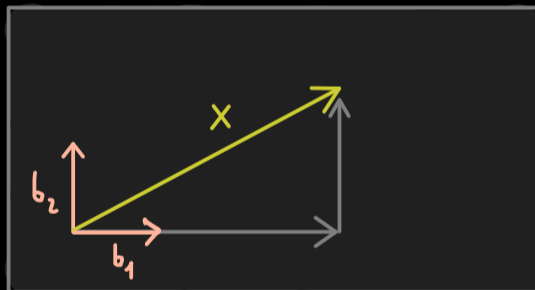
V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$,

$U \subseteq V$ k -dimensional subspace,

$\mathcal{B} = (b_1, b_2, \dots, b_k)$ ONB of U .

orthogonal projection: $x|_U = \sum_{j=1}^k b_j \underbrace{\langle b_j, x \rangle}_{\text{scalars}}$

The case $x \in U$:



$$x = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k$$

How to find?

↳ easy for ONB!

Result: V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ONB of U .

Then for each $u \in U$ we have the linear combination

$$u = \sum_{j=1}^k b_j \underbrace{\langle b_j, u \rangle}_{\substack{\in \mathbb{F} \\ \text{Fourier coefficients}}} \quad \left(\text{Fourier expansion of } u \text{ w.r.t. } \mathcal{B} \right)$$

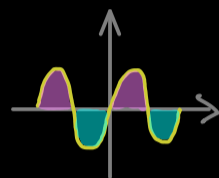
Example: $V = U = \text{Span}(x \mapsto \frac{1}{\sqrt{2}}, x \mapsto \cos(x), x \mapsto \cos(2x), x \mapsto \sin(x))$

(subspace in $\mathcal{F}(\mathbb{R})$)

with inner product: $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$

We get: $\langle x \mapsto \cos(x), x \mapsto \cos(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos(x))^2 dx = 1$

$\langle x \mapsto \cos(x), x \mapsto \sin(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\cos(x)\sin(x)}_{\text{odd function}} dx$



$= 0$

\vdots

$\implies \mathcal{B} = (x \mapsto \frac{1}{\sqrt{2}}, x \mapsto \cos(x), x \mapsto \cos(2x), x \mapsto \sin(x))$ ONB

Take u with $u(x) = (\sin(x))^2$ (actually $u \in V$)

Calculate: $\langle b_1, u \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} (\sin(x))^2 dx = \frac{1}{\sqrt{2}}$

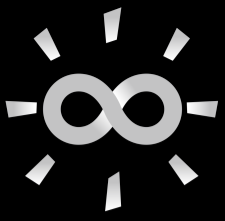
$\langle b_2, u \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) (\sin(x))^2 dx = \frac{1}{\pi} \frac{1}{3} (\sin(x))^3 \Big|_{-\pi}^{\pi} = 0$

$\langle b_3, u \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2x) (\sin(x))^2 dx \stackrel{\text{longer calculation}}{=} -\frac{1}{2}$

$\langle b_4, u \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin(x))^3 dx = 0$

$\implies u = b_1 \langle b_1, u \rangle + b_3 \langle b_3, u \rangle$

$(\sin(x))^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \cos(2x) \cdot \left(-\frac{1}{2}\right) = \frac{1}{2} \cdot (1 - \cos(2x))$



Abstract Linear Algebra - Part 20

V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

basis of U : (u_1, u_2, \dots, u_k) \rightsquigarrow ONB of U : (b_1, b_2, \dots, b_k)
Gram-Schmidt process/algorithm $\langle b_i, b_j \rangle = \delta_{ij}$

Gram-Schmidt orthonormalization:

(1) Normalize first vector:

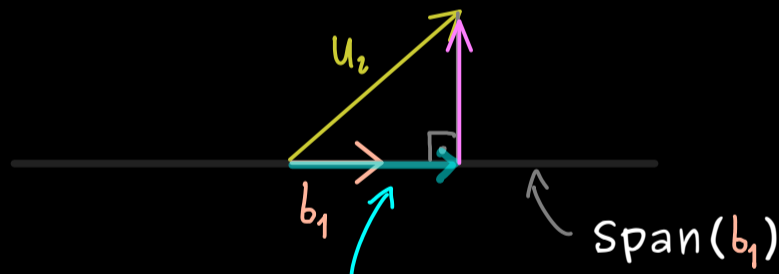
$$b_1 := \frac{1}{\|u_1\|} \cdot u_1$$

where

$$\|u_1\| := \sqrt{\langle u_1, u_1 \rangle}$$

length = 1?

(2) Next vector u_2 :

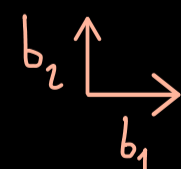


orthogonal projection of u_2 onto $\text{Span}(b_1)$:

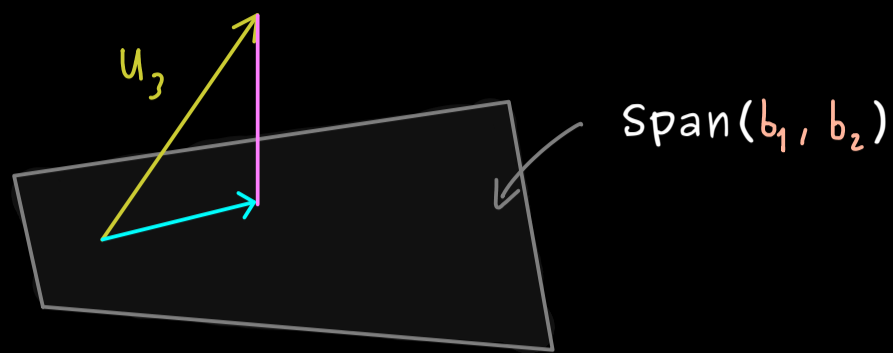
$$\hookrightarrow u_2|_{\text{Span}(b_1)} = b_1 \langle b_1, u_2 \rangle$$

normal component: $\tilde{b}_2 = u_2 - b_1 \langle b_1, u_2 \rangle$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$



(3) Next vector u_3 :



orthogonal projection of u_3 onto $\text{Span}(b_1, b_2)$:

$$\hookrightarrow u_3|_{\text{Span}(b_1, b_2)} := b_1 \langle b_1, u_3 \rangle + b_2 \langle b_2, u_3 \rangle$$

normal component: $\tilde{b}_3 = u_3 - b_1 \langle b_1, u_3 \rangle - b_2 \langle b_2, u_3 \rangle$

normalize it: $b_3 := \frac{1}{\|\tilde{b}_3\|} \tilde{b}_3$

•
• continue!
•

(k) Next vector u_k :

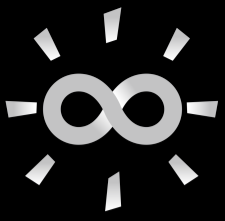


orthogonal projection of u_k onto $\text{Span}(b_1, b_2, \dots, b_{k-1})$

$$\hookrightarrow u_k|_{\text{Span}(b_1, b_2, \dots, b_{k-1})} := \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$$

normal component: $\tilde{b}_k = u_k - \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$

normalize it: $b_k := \frac{1}{\|\tilde{b}_k\|} \tilde{b}_k \implies \text{ONB of } U : (b_1, b_2, \dots, b_k)$



Abstract Linear Algebra - Part 21

V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

basis of U : (u_1, u_2, \dots, u_k) $\xrightarrow{\text{Gram-Schmidt process/algorithm}}$ ONB of U : (b_1, b_2, \dots, b_k)

Example: $V = \mathcal{P}([-1, 1], \mathbb{R})$ polynomial space with inner product:

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$$

Take $U = \mathcal{P}_2([-1, 1], \mathbb{R})$ with basis (m_0, m_1, m_2) $m_0: x \mapsto 1$
 $m_1: x \mapsto x$
 $m_2: x \mapsto x^2$
(polynomials of degree ≤ 2) not ONB!

Gram-Schmidt orthonormalization:

(1) Normalize first vector: $\|m_0\|^2 = \langle m_0, m_0 \rangle = \int_{-1}^1 1 \cdot 1 dx = 2$

$$b_0 := \frac{1}{\|m_0\|} \cdot m_0 = \frac{1}{\sqrt{2}} m_0, \quad b_0(x) = \frac{1}{\sqrt{2}}$$

(2) Next vector m_1 :

normal component: $\tilde{b}_1 = m_1 - b_0 \langle b_0, m_1 \rangle = m_1$
 $\underbrace{\langle b_0, m_1 \rangle}_{= \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot x dx = 0}$

normalize it: $b_1 := \frac{1}{\|\tilde{b}_1\|} \tilde{b}_1, \quad \|\tilde{b}_1\|^2 = \int_{-1}^1 x \cdot x dx = \frac{1}{3} x^3 \Big|_{-1}^1 = \frac{2}{3}$

$$= \sqrt{\frac{3}{2}} m_1, \quad b_1(x) = \sqrt{\frac{3}{2}} x$$

(3) Next vector m_2 :

normal component:

$$\begin{aligned}\tilde{b}_2 &= m_2 - b_0 \langle b_0, m_2 \rangle - b_1 \langle b_1, m_2 \rangle \\ &= m_2 - \underbrace{b_0 \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot x^2 dx}_{= \frac{1}{\sqrt{2}} \cdot \frac{1}{3} x^3 \Big|_{-1}^1 = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3}} - \underbrace{b_1 \int_{-1}^1 \sqrt{\frac{3}{2}} x \cdot x^2 dx}_{= 0} \\ &= m_2 - \frac{1}{3} m_0, \quad \tilde{b}_2(x) = x^2 - \frac{1}{3}\end{aligned}$$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$, $\|\tilde{b}_2\|^2 = \int_{-1}^1 (x^2 - \frac{1}{3})(x^2 - \frac{1}{3}) dx$

$$\begin{aligned}&= \int_{-1}^1 (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx \\ &= \frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x \Big|_{-1}^1 = \frac{8}{45}\end{aligned}$$

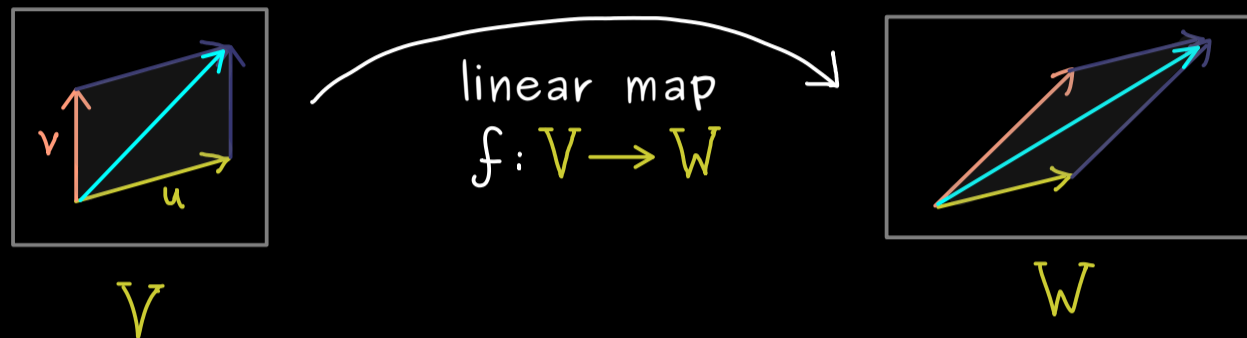
$$\Rightarrow b_2(x) = \sqrt{\frac{45}{8}} \cdot (x^2 - \frac{1}{3})$$

\rightsquigarrow ONB for $\mathcal{P}_2([-1, 1], \mathbb{R})$

(Legendre polynomials)



Abstract Linear Algebra - Part 22



Recall: $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear \iff matrix $A \in \mathbb{R}^{m \times n}$

Definition: Let V, W be two \mathbb{F} -vector spaces. (same \mathbb{F} for both)

A map $f: V \rightarrow W$ is called linear if:

$$(1) f(u+v) = f(u) + f(v)$$

vector addition in V

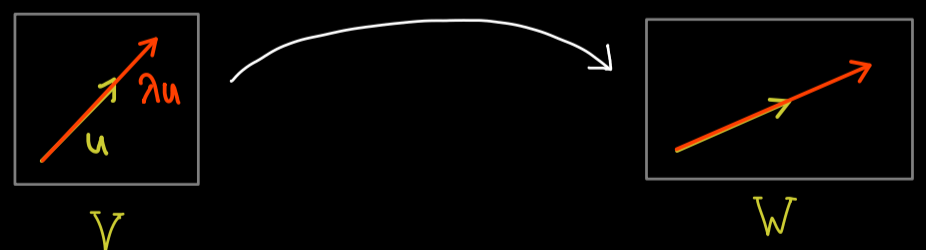
vector addition in W

$$(2) f(\lambda \cdot u) = \lambda \cdot f(u)$$

scalar multiplication in V

scalar multiplication in W

for all $u, v \in V, \lambda \in \mathbb{F}$.



Remember: $f(0_V) = f(0 \cdot u) \stackrel{(2)}{=} 0 \cdot f(u) = 0_W$

Example: (a) $V = \mathbb{F}^3$, $W = \mathbb{F}$, $a \in V$.

$f(u) := \langle a, u \rangle_{\text{standard}}$ is a linear map.

$\equiv a^* u$ (matrix multiplication)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^* = (\overline{a_1} \ \overline{a_2} \ \overline{a_3})$$

(transpose + complex conjugation)

(b) $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$

$$l: V \rightarrow W$$

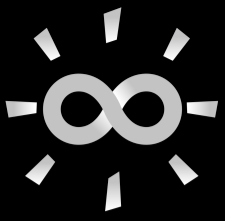
$$p \mapsto p'$$

$$l(x \mapsto x^2) = x \mapsto 2x$$

is a linear map!

$$l(p+q) = (p+q)' = p' + q' = l(p) + l(q)$$

$$l(\lambda p) = (\lambda p)' = \lambda p' = \lambda l(p)$$



Abstract Linear Algebra - Part 23

Recall: linear map or linear operator $l: V \rightarrow W$:

$$l(x+y) = l(x) + l(y)$$

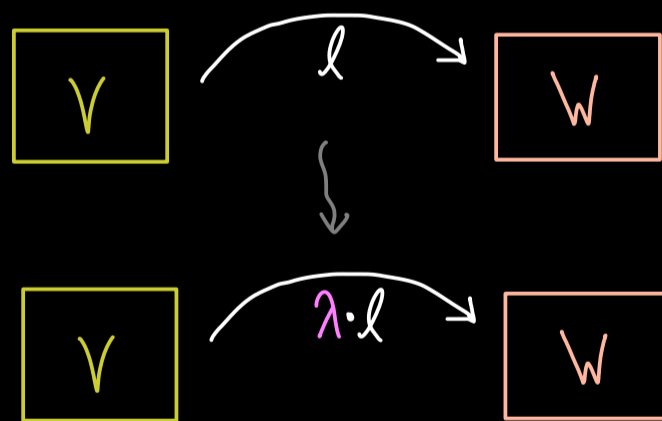
$$l(\lambda \cdot x) = \lambda \cdot l(x)$$

Definition: Let V, W be two \mathbb{F} -vector spaces. (same \mathbb{F} for both)

For $k: V \rightarrow W$, $l: V \rightarrow W$ linear maps, we define:

$$k+l: V \rightarrow W, \quad (k+l)(x) := k(x) + l(x)$$

$$\text{(given } \lambda \in \mathbb{F} \text{)} \quad \lambda \cdot l: V \rightarrow W, \quad (\lambda \cdot l)(x) := \lambda \cdot l(x)$$



Result: With $+$, \cdot from above, the set $\mathcal{L}(V, W) = \{ l: V \rightarrow W \mid \text{linear} \}$

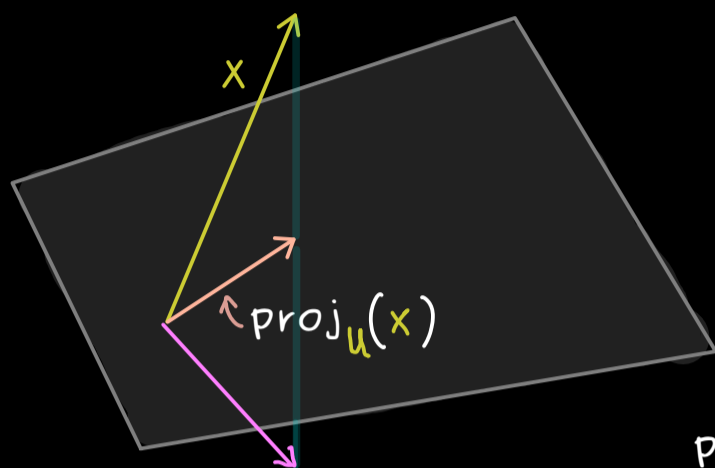
forms an \mathbb{F} -vector space.

Zero vector $0 \in \mathcal{L}(V, W)$ is given by the zero map $0(x) = 0_W$ for all $x \in V$.
↑ zero vector in W

Example: V with inner product $\langle \cdot, \cdot \rangle$ and ONB (e_1, e_2, \dots, e_n) .

$$U = \text{span}(e_1, e_2, \dots, e_{n-1})$$

Orthogonal projection onto U : $\text{proj}_U: V \rightarrow V$



$$x \mapsto \sum_{j=1}^{n-1} e_j \langle e_j, x \rangle$$

linear map

$\text{proj}_{U^\perp}: V \rightarrow V$

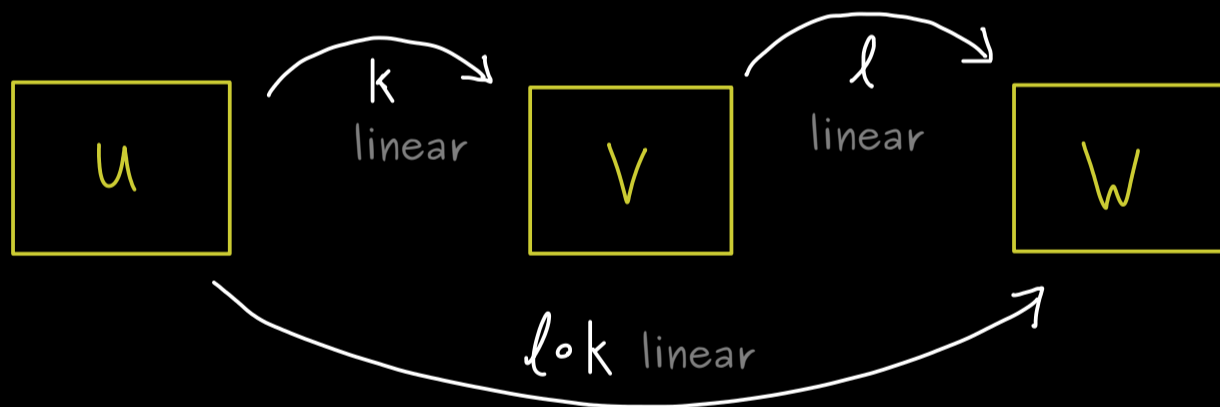
$$x \mapsto e_n \langle e_n, x \rangle$$

linear map

Addition: $\text{proj}_U + \text{proj}_{U^\perp} = \text{id}_V$

Subtraction: $\text{proj}_U - \text{proj}_{U^\perp} = \text{id}_V - 2 \cdot \text{proj}_{U^\perp}$ reflection

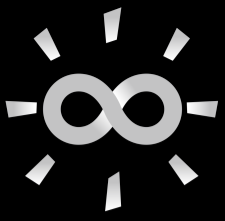
Composition:



$$k \in \mathcal{L}(U, V), l \in \mathcal{L}(V, W) \Rightarrow l \circ k \in \mathcal{L}(U, W)$$

Example: $\text{proj}_U \circ \text{proj}_U = \text{proj}_U$, $\text{proj}_U \circ \text{proj}_{U^\perp} = 0$

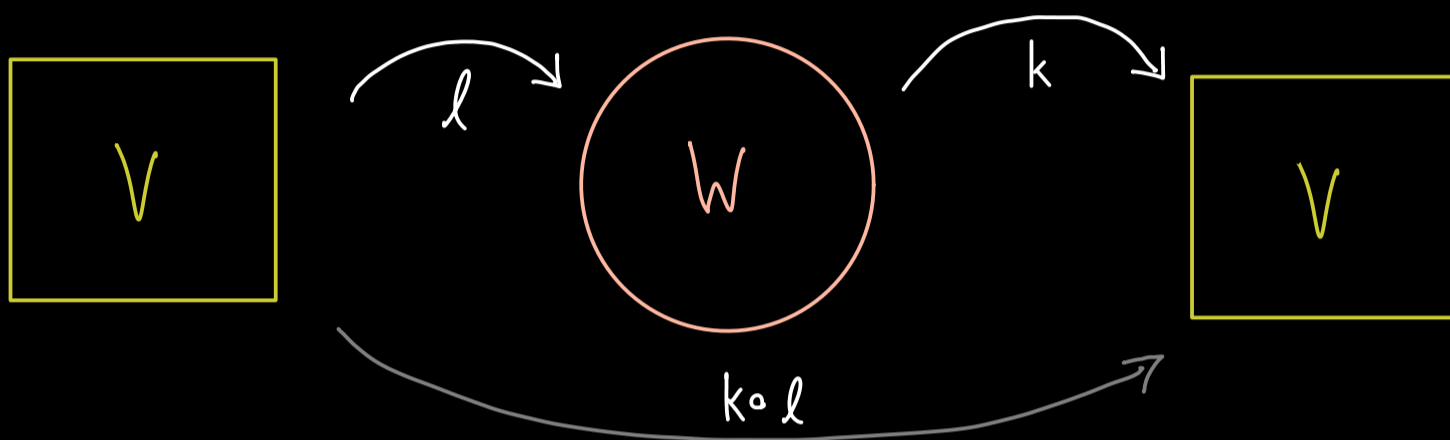
zero vector in $\mathcal{L}(V, V)$



Abstract Linear Algebra - Part 24

$l: V \rightarrow W$ linear map preserves the structure of the vector space.

\Leftrightarrow
(vector space) homomorphism



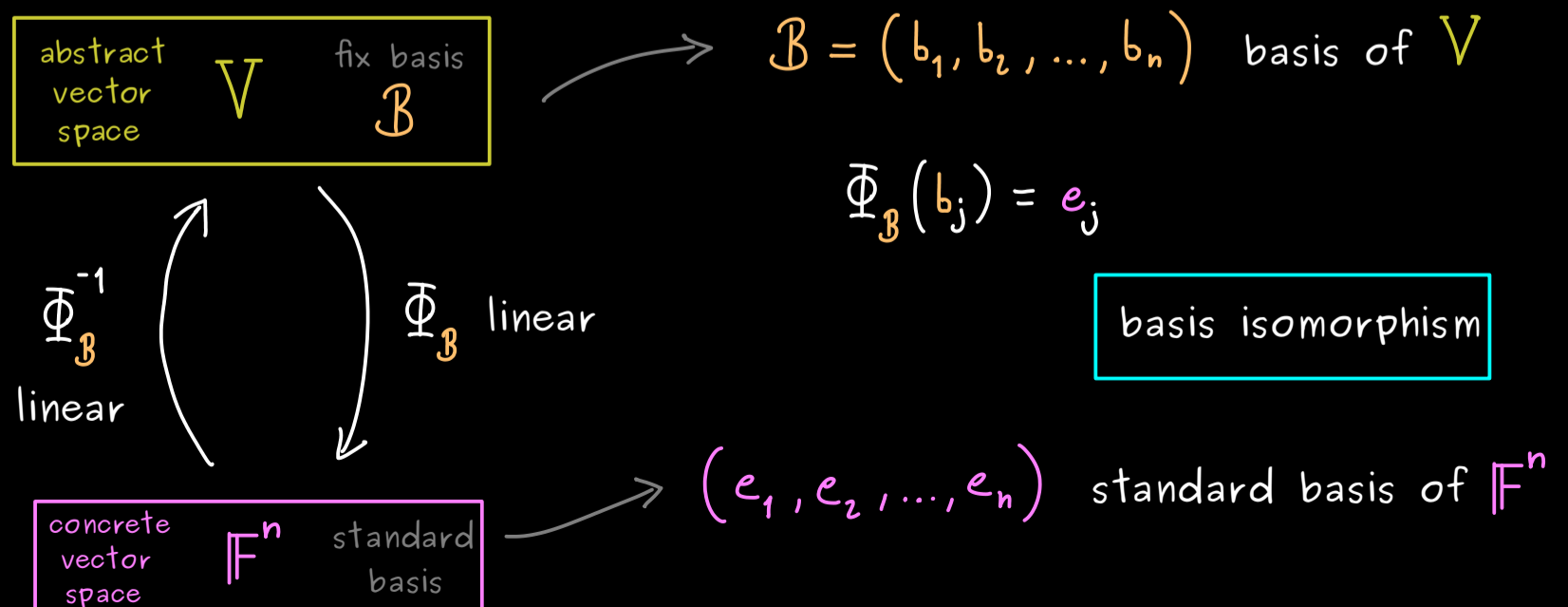
Reminder: (just maps on sets) $f: V \rightarrow W$ is called invertible if there is a map $g: W \rightarrow V$ with $g \circ f = id_V$ and $f \circ g = id_W$
 \rightarrow denoted by f^{-1}

f bijective $\Leftrightarrow f$ invertible

Fact: $l: V \rightarrow W$ linear + bijective $\Rightarrow l^{-1}: W \rightarrow V$ linear

(see part 31 in "Linear Algebra")

Example:



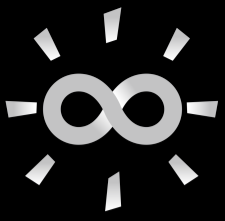
Definition: $\ell: V \rightarrow W$ homomorphism + $\ell^{-1}: W \rightarrow V$ homomorphism

\rightsquigarrow is called an isomorphism

Remember: (vector space) isomorphism = bijective linear map

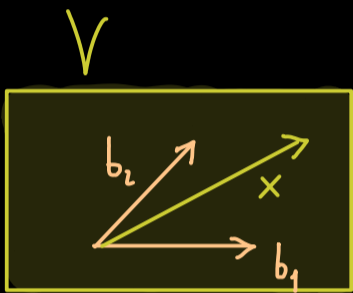
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linear isomorphism



Abstract Linear Algebra - Part 25

$$\begin{aligned} \ell: V \longrightarrow W \quad \text{linear:} \quad & \ell(x+y) = \ell(x) + \ell(y) \\ & \ell(\lambda \cdot x) = \lambda \cdot \ell(x) \end{aligned}$$



$\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V

$\implies x \in V$ can be written as $x = \alpha_1 b_1 + \dots + \alpha_n b_n$

Hence:
$$\begin{aligned} \ell(x) &= \ell(\alpha_1 b_1 + \dots + \alpha_n b_n) \\ &= \alpha_1 \ell(b_1) + \alpha_2 \ell(b_2) + \dots + \alpha_n \ell(b_n) \end{aligned}$$

If know these, we know ℓ

Example: $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$, $\ell: V \rightarrow W$
 $p \mapsto p'$
is a linear map!

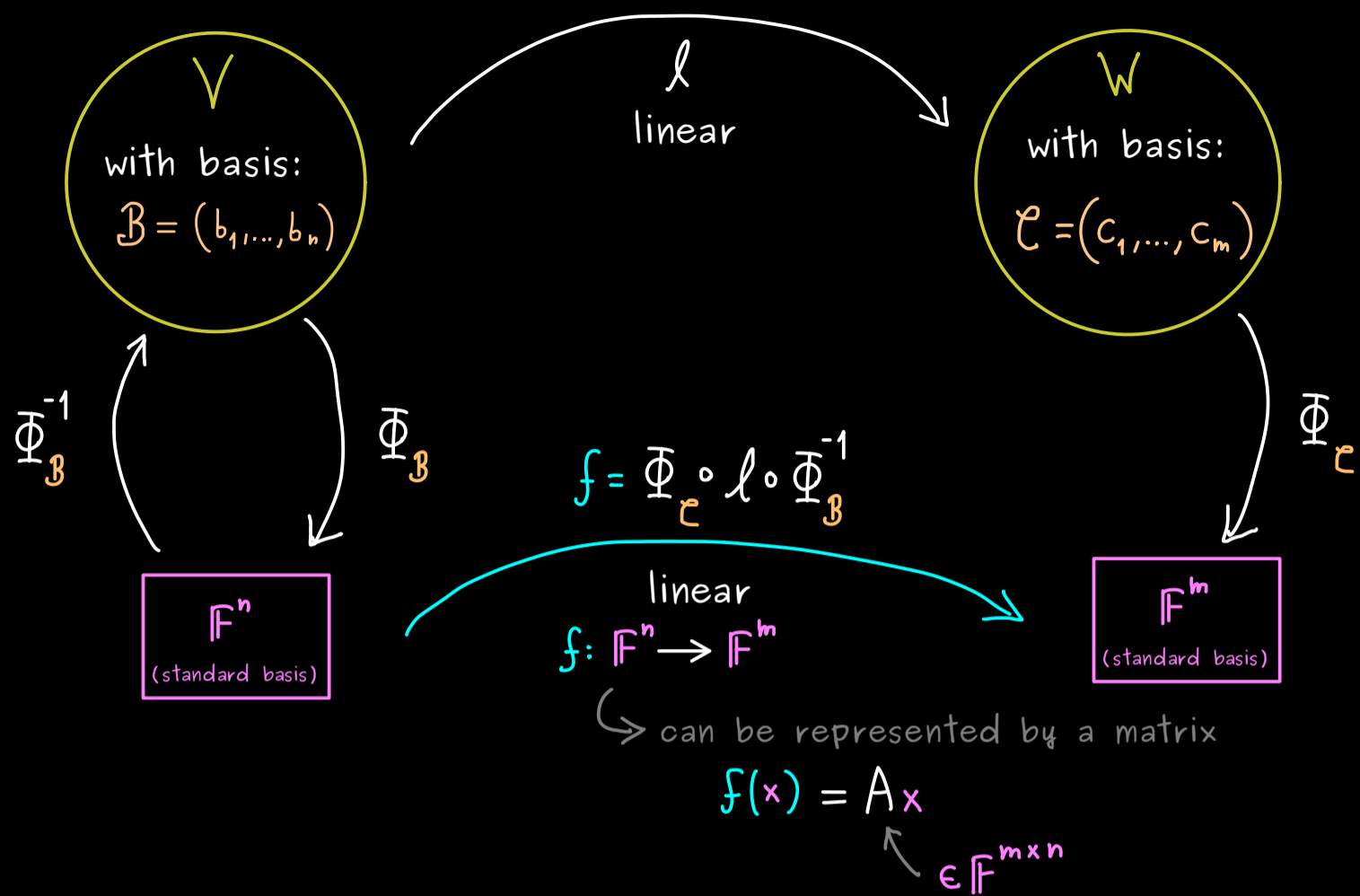
basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$$\ell(m_0) = 0 \leftarrow \text{zero vector: } x \mapsto 0$$

$$\ell(m_k) = k \cdot m_{k-1}, \quad k \in \{1, 2, 3\}$$

Result:



First column of A : $f(e_1) = (\Phi_C \circ l \circ \Phi_B^{-1})(e_1) = \Phi_C(l(b_1))$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$

Matrix representation: For a linear map $l: V \rightarrow W$,

$$l_{\mathcal{C} \leftarrow \mathcal{B}} := \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(l(b_1)) & \Phi_{\mathcal{C}}(l(b_2)) & \dots & \Phi_{\mathcal{C}}(l(b_n)) \\ | & | & & | \end{pmatrix} \in \mathbb{F}^{m \times n}$$

is called the matrix representation of l with respect to \mathcal{B} and \mathcal{C} .

Example (from before)

$V = \mathcal{P}_3(\mathbb{R})$ basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1, m_k: x \mapsto x^k$

$l: V \rightarrow W$
 $p \mapsto p'$

$W = \mathcal{P}_2(\mathbb{R})$ basis: $\mathcal{C} = (c_1, c_2, c_3) = (m_0, m_1, m_2)$

is a linear map:

$\Phi_{\mathcal{C}}(l(b_1)) = \Phi_{\mathcal{C}}(l(m_0)) = \Phi_{\mathcal{C}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$

$\Phi_{\mathcal{C}}(l(b_2)) = \Phi_{\mathcal{C}}(l(m_1)) = \Phi_{\mathcal{C}}(m_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$

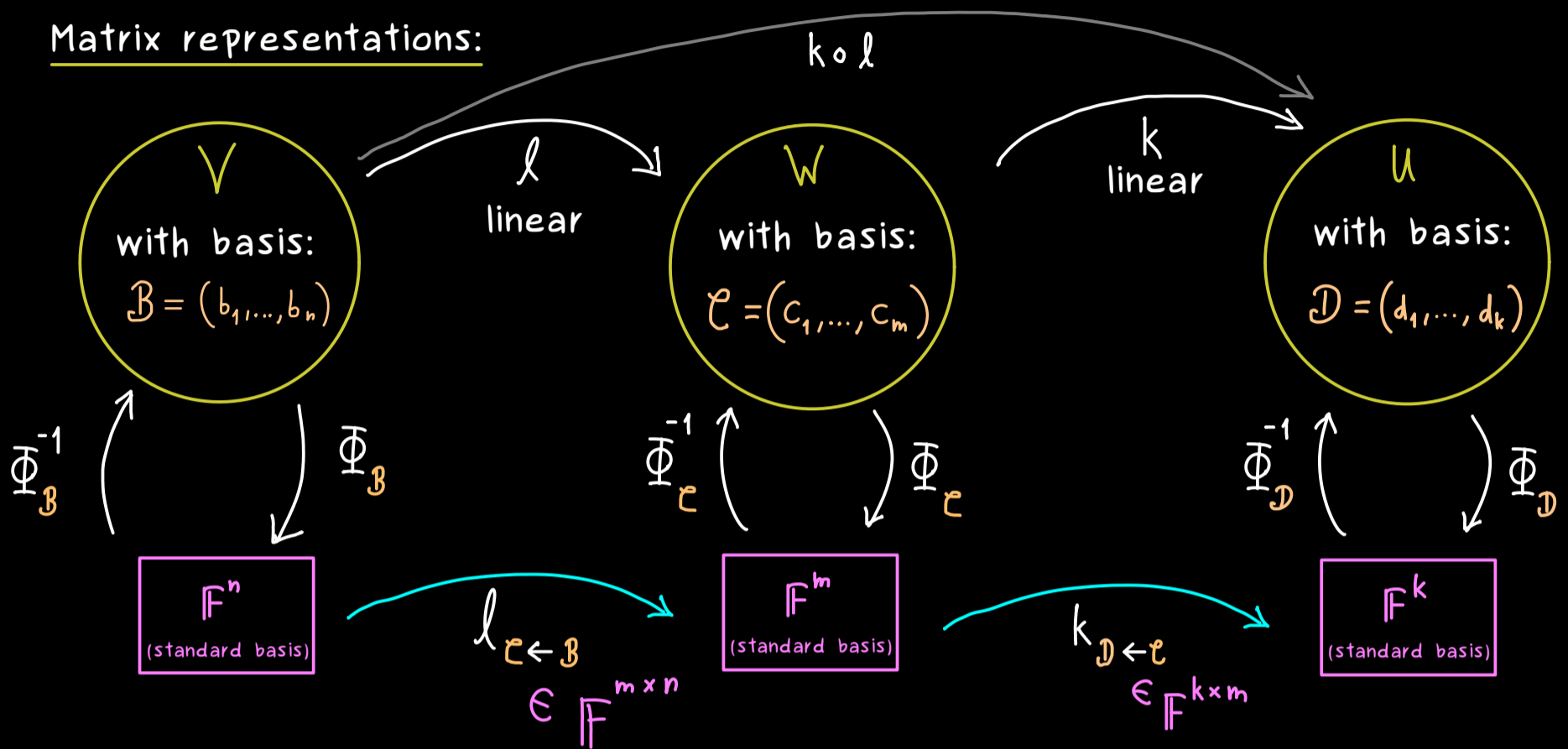
⋮

$\Rightarrow l_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ matrix representation of l



Abstract Linear Algebra - Part 26

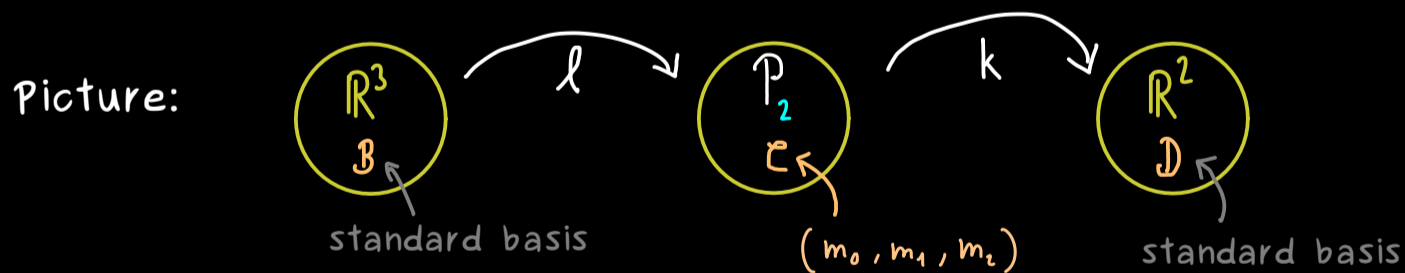
Matrix representations:



We get: $(k \circ l)_{D \leftarrow B} = k_{D \leftarrow C} l_{C \leftarrow B}$ (matrix product)

Example: $l: \mathbb{R}^3 \rightarrow \mathcal{P}_2(\mathbb{R})$, $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto (v_1 + v_2 + v_3) \cdot m_0 + (v_1 + v_2) \cdot m_1 + v_1 \cdot m_2$
with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$k: \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}^2$, $p \mapsto \begin{pmatrix} p'(1) \\ p(1) - p''(1) \end{pmatrix}$



$(k \circ l)_{D \leftarrow B} = ?$

$$l_{C \leftarrow B} = \begin{pmatrix} | & | & | \\ \Phi_C(l(b_1)) & \Phi_C(l(b_2)) & \Phi_C(l(b_3)) \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

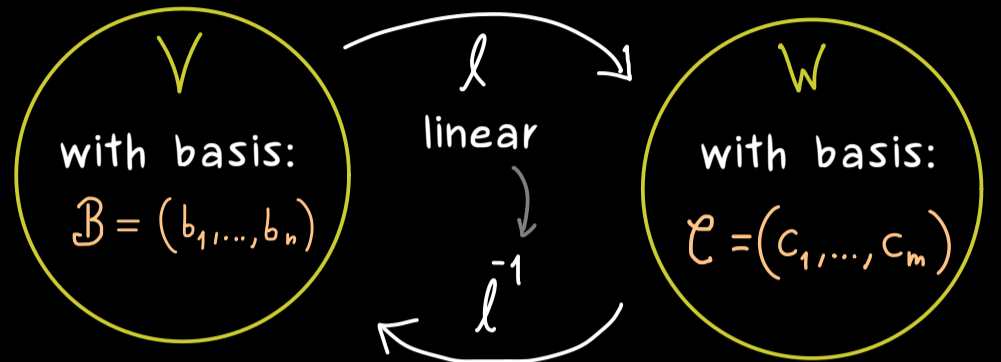
$$k_{D \leftarrow C} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

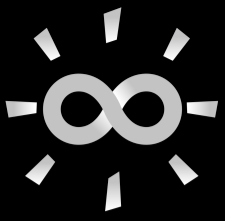
$$(k \circ l)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} l_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Corollary:

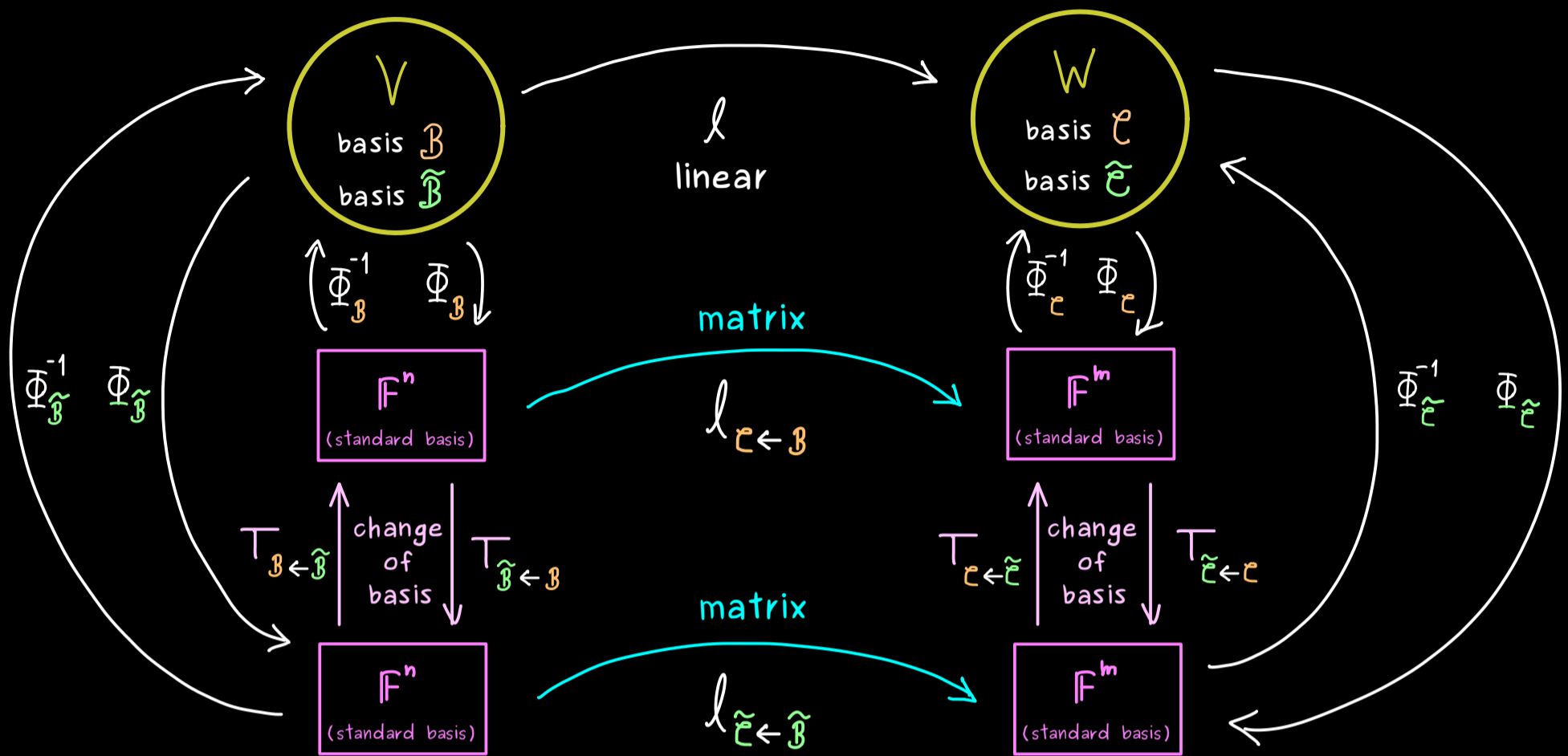
$$(l^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = (l_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}$$

$n = m$





Abstract Linear Algebra - Part 27



Result:

$$l_{\tilde{C} \leftarrow \tilde{B}} = T_{\tilde{C} \leftarrow C} l_{C \leftarrow B} T_{B \leftarrow \tilde{B}}$$

Example: $l: \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$, $l(p) = p'$ linear map!

$$B = (m_3, m_2, m_1, m_0) \quad C = (m_2, m_1, m_0)$$

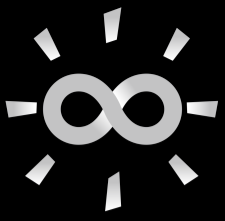
$$\tilde{B} = (2m_3 - m_1, m_2 + m_0, m_1 + m_0, m_1 - m_0), \quad \tilde{C} = (m_2 - \frac{1}{2}m_1, m_2 + \frac{1}{2}m_1, m_0)$$

matrix representation: $l_{C \leftarrow B} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

change-of-basis matrices: $T_{B \leftarrow \tilde{B}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

$$T_{\mathcal{E} \leftarrow \tilde{\mathcal{E}}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{inverse}} T_{\tilde{\mathcal{E}} \leftarrow \mathcal{E}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} l_{\tilde{\mathcal{E}} \leftarrow \hat{\mathcal{B}}} &= T_{\tilde{\mathcal{E}} \leftarrow \mathcal{E}} l_{\mathcal{E} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \hat{\mathcal{B}}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} \end{aligned}$$



Abstract Linear Algebra - Part 28

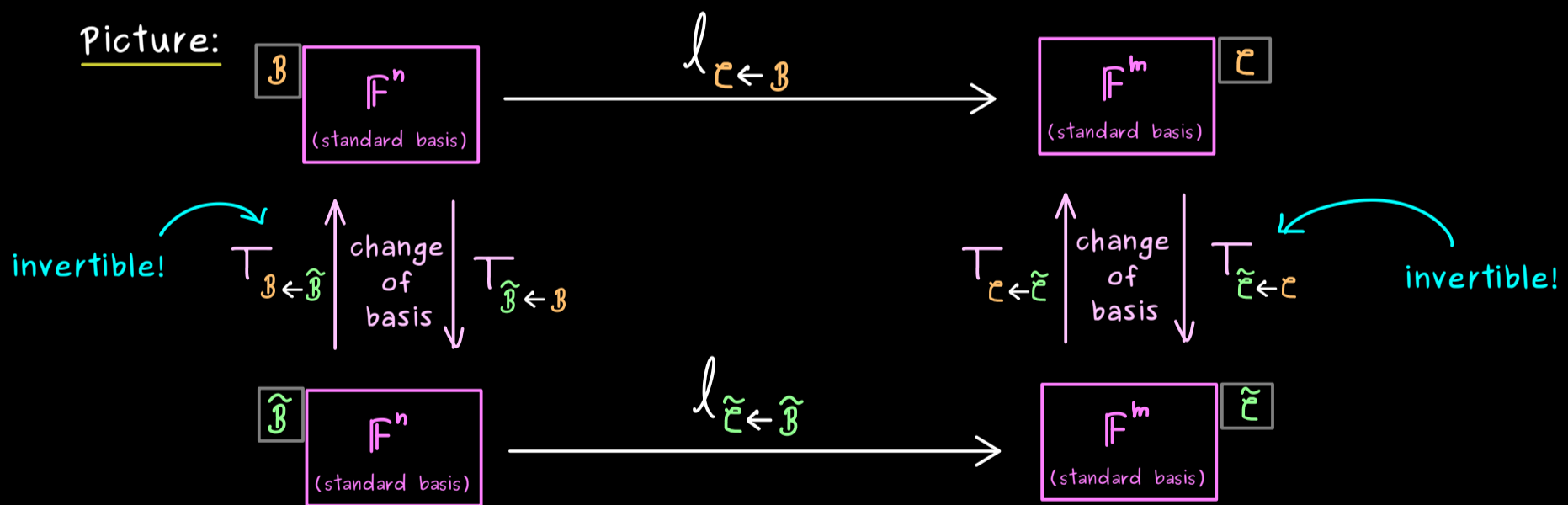
Fact: $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$ are different but

they describe the same linear map $\ell: \mathcal{P}_3(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R})$, $\ell(p) = p'$ with respect to different bases.

Question: $\ell: V \rightarrow W$ linear, $A = \ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}$.

For another $\tilde{A} \in \mathbb{F}^{m \times n}$, can we find bases such that $\tilde{A} = \ell_{\tilde{\mathcal{C}} \leftarrow \tilde{\mathcal{B}}}$?

If **YES!**, then we say A and \tilde{A} are equivalent.



Definition: A matrix $\tilde{A} \in \mathbb{F}^{m \times n}$ is called equivalent to a matrix $A \in \mathbb{F}^{m \times n}$

if there are invertible matrices $S \in \mathbb{F}^{m \times m}$, $T \in \mathbb{F}^{n \times n}$, such that:

$$\tilde{A} = S A T.$$

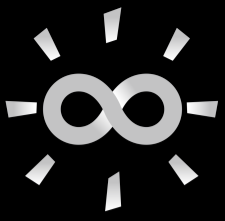
We write: $\tilde{A} \sim A$

Remark: \sim defines an equivalence relation on $\mathbb{F}^{m \times n}$:

(1) reflexive: $A \sim A$ for all $A \in \mathbb{F}^{m \times n}$

(2) symmetric: $A \sim B \Rightarrow B \sim A$ for all $A, B \in \mathbb{F}^{m \times n}$

(3) transitive: $A \sim B \wedge B \sim C \Rightarrow A \sim C$ for all $A, B, C \in \mathbb{F}^{m \times n}$



Abstract Linear Algebra - Part 29

Equivalence relation: $A, B \in \mathbb{F}^{m \times n}$, $A \sim B$ \leftarrow they both represent the same linear map $\ell: V \rightarrow W$
 \leftarrow there are invertible matrices S, T with $B = S A T$.

kernel and range?

$$\text{Ker}(B) = \text{Ker}(S A T) = \left\{ x \in \mathbb{F}^n \mid \underbrace{A T x}_{\in \text{Ker}(A)} = 0 \right\} = T^{-1} \text{Ker}(A)$$

$$\begin{aligned} \text{Ran}(B) &= \text{Ran}(S A T) = \left\{ S A T x \mid x \in \mathbb{F}^n \right\} \\ &= \left\{ S A \tilde{x} \mid \underbrace{\tilde{x} \in \mathbb{F}^n}_{\in \text{Ran}(A)} \right\} = S \text{Ran}(A) \end{aligned}$$

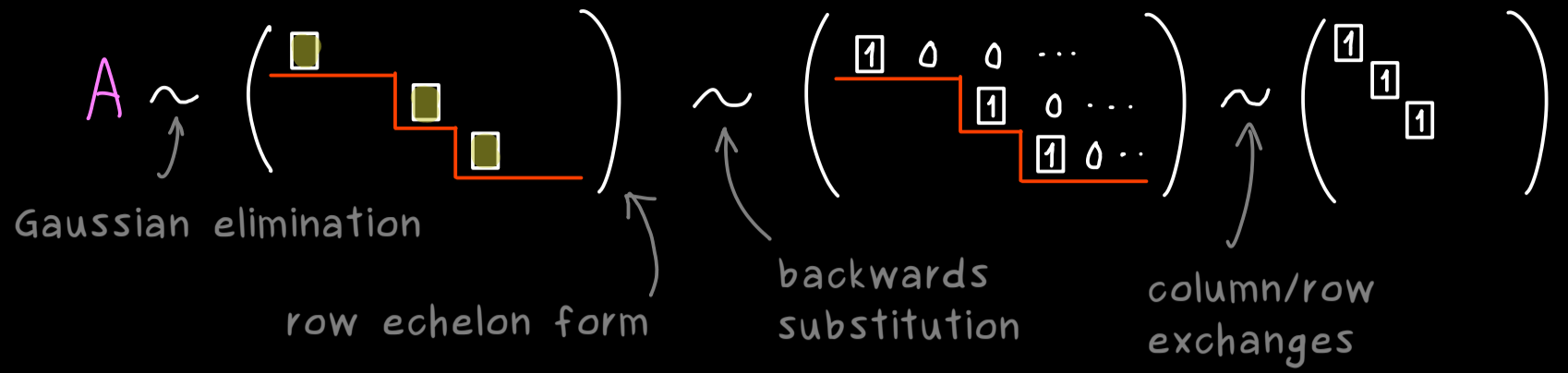
Result: $A \sim B \implies$

$$\begin{array}{ccc} \text{rank}(A) & = & \text{rank}(B) \\ + & & + \\ \text{nullity}(A) & = & \text{nullity}(B) \\ \parallel & & \parallel \\ n & & n \end{array}$$

Proposition: For $A, B \in \mathbb{F}^{m \times n}$, we have:

$$A \sim B \iff \text{rank}(A) = \text{rank}(B)$$

Proof:



$$\Rightarrow A \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}$$

with $r = \text{rank}(A)$

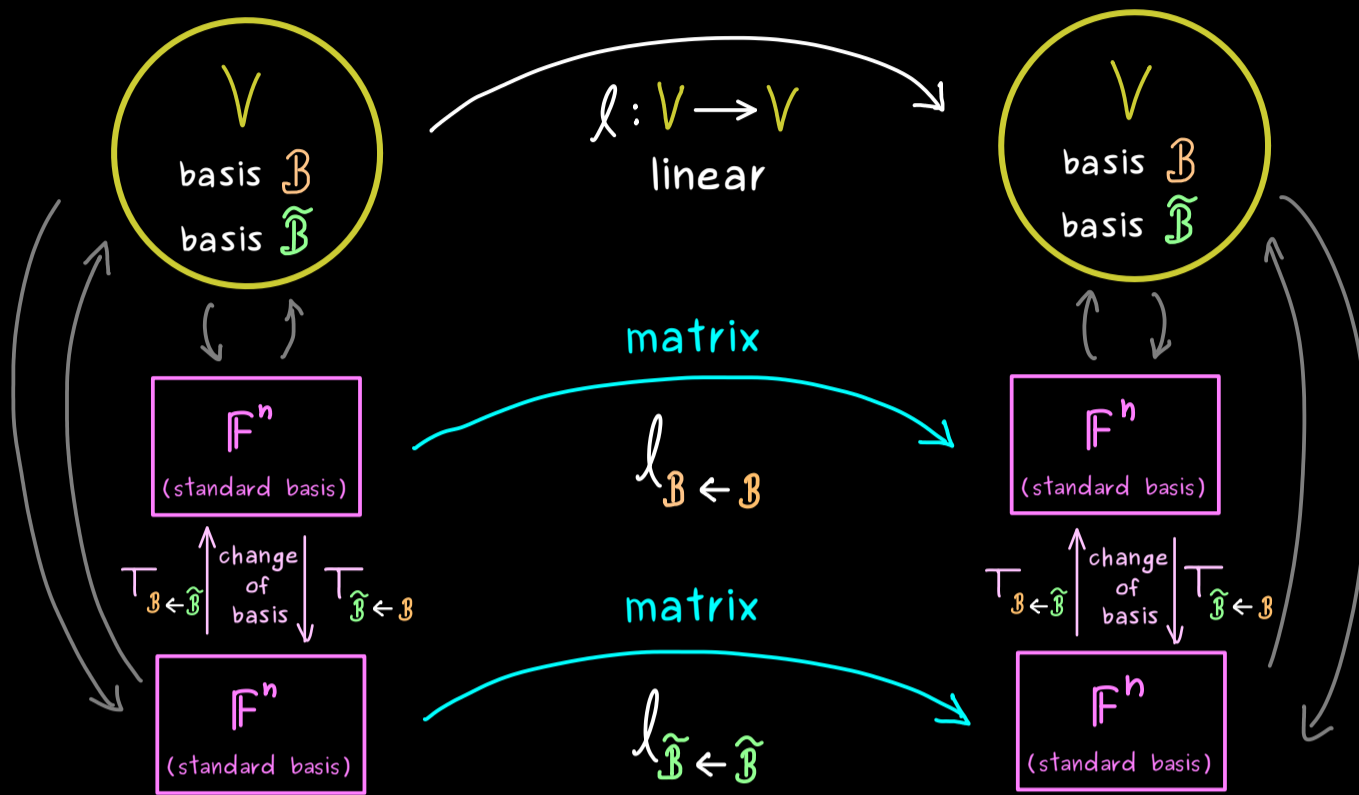
$$B \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}$$

$r = \text{rank}(B)$

□



Abstract Linear Algebra - Part 30



We have:

$$l_{\tilde{\mathcal{B}} \leftarrow \tilde{\mathcal{B}}} = T_{\tilde{\mathcal{B}} \leftarrow \mathcal{B}} l_{\mathcal{B} \leftarrow \mathcal{B}} T_{\mathcal{B} \leftarrow \tilde{\mathcal{B}}}$$

$$\parallel \quad \parallel \quad \parallel \quad \parallel$$

$$\tilde{A} = T^{-1} A T$$

Definition: A matrix $\tilde{A} \in \mathbb{F}^{n \times n}$ is called similar to a matrix $A \in \mathbb{F}^{n \times n}$ if there is an invertible $T \in \mathbb{F}^{n \times n}$ such that:

$$\tilde{A} = T^{-1} A T.$$

We write: $\tilde{A} \approx A$.

Remark: \approx defines an equivalence relation on $\mathbb{F}^{n \times n}$:

(1) reflexive: $A \approx A$ for all $A \in \mathbb{F}^{n \times n}$

(2) symmetric: $A \approx B \Rightarrow B \approx A$ for all $A, B \in \mathbb{F}^{n \times n}$

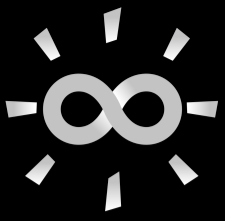
(3) transitive: $A \approx B \wedge B \approx C \Rightarrow A \approx C$ for all $A, B, C \in \mathbb{F}^{n \times n}$

Easy to see: $A \approx B \Rightarrow A \sim B$

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not\approx \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\hookrightarrow T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

\approx is characterized by the so-called Jordan normal form



Abstract Linear Algebra - Part 31

$l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces (finite-dimensional).

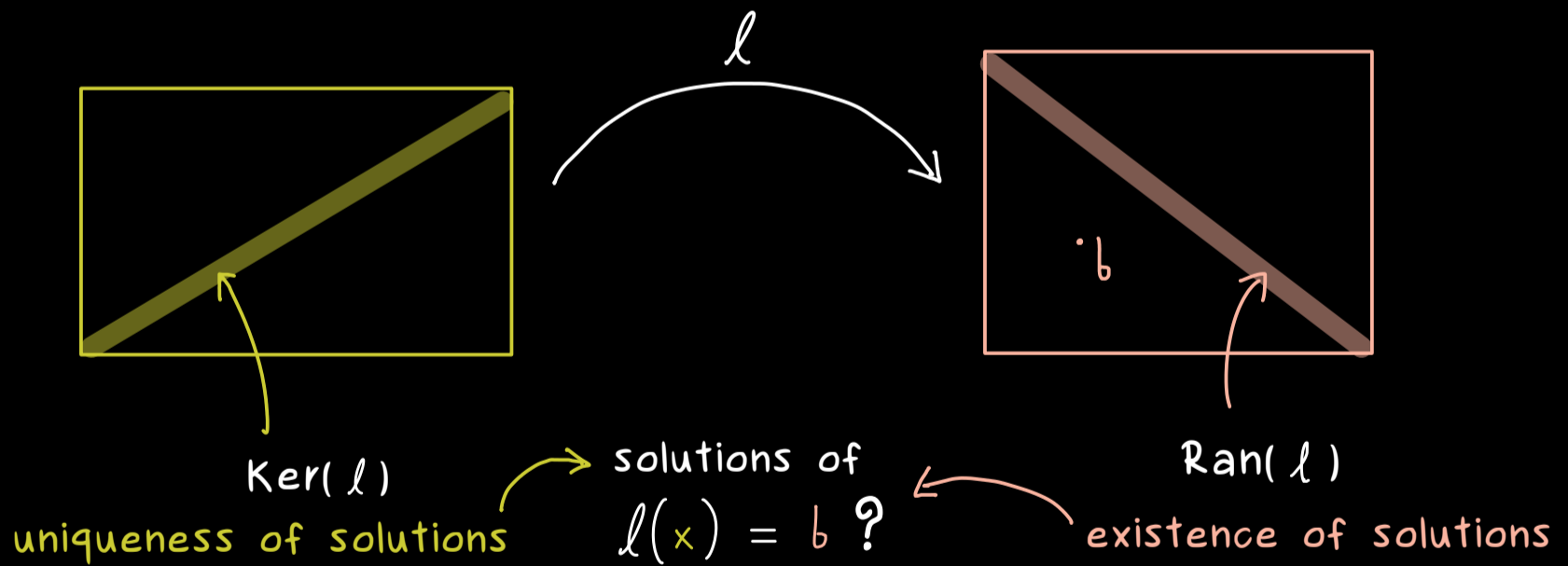
For $b \in W$:

$$l(x) = b \quad \text{solutions } x \in V$$

matrix representation

$$l_{\mathcal{C} \leftarrow \mathcal{B}} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \quad \left(\begin{array}{l} \text{system of} \\ \text{linear equations} \end{array} \right)$$

Definition: $\text{Ker}(l) := \{x \in V \mid l(x) = 0\}$ kernel of the linear map l
 $\text{Ran}(l) := \{w \in W \mid \text{there is } x \in V \text{ with } l(x) = w\}$ range of l



Proposition: $l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces, $b \in W$.

The solution set $S := \{x \in V \mid l(x) = b\}$

is either empty or an affine subspace: $S = \emptyset$ or

$$S = x_0 + \text{Ker}(l) \quad (\text{with } x_0 \in V)$$

Proof: Assume $x_0 \in S$ ($l(x_0) = b$).

Take any $v \in V$ and look at $x_0 + v$:

$$\begin{aligned} x_0 + v \in S &\iff l(x_0 + v) = b \iff \overset{\text{linear map}}{l(x_0)} + \overset{b}{l(v)} = b \\ &\iff l(v) = 0 \iff v \in \text{Ker}(l) \quad \square \end{aligned}$$

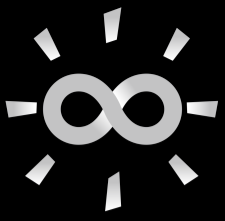
Rank-nullity theorem: $l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces (finite-dimensional)

$$\dim(\text{Ran}(l)) + \dim(\text{Ker}(l)) = \dim(V)$$

$$\parallel \text{part 28/29} \quad \parallel \quad \parallel$$

with matrix representations

$$\rightsquigarrow \dim(\text{Ran}(l_{\mathcal{C} \leftarrow \mathcal{B}})) + \dim(\text{Ker}(l_{\mathcal{C} \leftarrow \mathcal{B}})) = n$$



Abstract Linear Algebra - Part 32

$l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces

$$\dim(\text{Ran}(l)) + \dim(\text{Ker}(l)) = \dim(V)$$

\rightsquigarrow helps for solving linear equation $l(x) = b$

Example: $V = W = \mathcal{P}_3(\mathbb{R})$ together with monomial basis $(m_3, m_2, m_1, m_0) =: \mathcal{B}$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$$l: V \rightarrow W$$

$$p \mapsto p' \quad \Rightarrow \quad l(m_k) = k \cdot m_{k-1}, \quad l(m_0) = 0$$

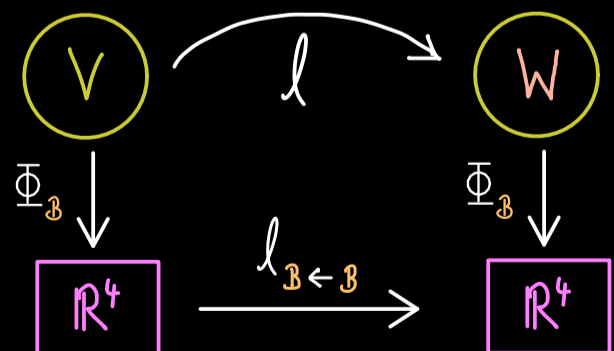
matrix representation: $l_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\text{Ker}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\text{Ran}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span} \left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Recall general picture:

$$l = \Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}$$



$$\begin{aligned} \text{Ker}(l) &= \text{Ker}(\underbrace{\Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}}_{\text{arrow}}) \\ &= \Phi_{\mathcal{B}}^{-1} \text{Ker}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \text{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \text{Span}(m_0) \end{aligned}$$

$$\begin{aligned} \text{Ran}(l) &= \text{Ran}(\underbrace{\Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}}_{\text{arrow}}) \\ &= \Phi_{\mathcal{B}}^{-1} \text{Ran}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \text{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \text{Span}(m_2, m_1, m_0) \end{aligned}$$

Linear equation: $l(p) = g$? solutions give antiderivatives/primitives for g

$$\Rightarrow \mathcal{S}^1 = \emptyset \quad \text{or} \quad \mathcal{S}^1 = \tilde{p} + \text{Ker}(l) \quad \text{with} \quad \tilde{p}' = g$$