

Abstract Linear Algebra - Part 32

$l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces

$$\dim(\text{Ran}(l)) + \dim(\text{Ker}(l)) = \dim(V)$$

\rightsquigarrow helps for solving linear equation $l(x) = b$

Example: $V = W = \mathcal{P}_3(\mathbb{R})$ together with monomial basis $(m_3, m_2, m_1, m_0) =: \mathcal{B}$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$$l: V \rightarrow W$$

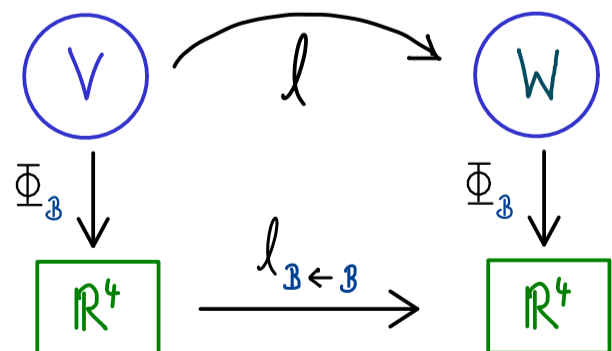
$$p \mapsto p' \quad \Rightarrow \quad l(m_k) = k \cdot m_{k-1}, \quad l(m_0) = 0$$

matrix representation: $l_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\text{Ker}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\text{Ran}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \text{Span} \left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Recall general picture: $l = \Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}$



$$\begin{aligned} \text{Ker}(l) &= \text{Ker}\left(\underbrace{\Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}}_{\leftarrow}\right) \\ &= \Phi_{\mathcal{B}}^{-1} \text{Ker}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \text{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \text{Span}(m_0) \end{aligned}$$

$$\begin{aligned} \text{Ran}(l) &= \text{Ran}\left(\underbrace{\Phi_{\mathcal{B}}^{-1} \circ l_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}}_{\leftarrow}\right) \\ &= \Phi_{\mathcal{B}}^{-1} \text{Ran}(l_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \text{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \text{Span}(m_2, m_1, m_0) \end{aligned}$$

Linear equation: $l(p) = g$? solutions give antiderivatives/primitives for g

$$\Rightarrow \mathcal{S} = \emptyset \quad \text{or} \quad \mathcal{S} = \tilde{p} + \text{Ker}(l) \quad \text{with} \quad \tilde{p}' = g$$