The Bright Side of Mathematics

Abstract Linear Algebra - Part 32

$\ell : V \longrightarrow W$ linear, V, W $\mathbb{F}$ - vector spaces

$$\dim(\operatorname{Ran}(\ell)) + \dim(\operatorname{Ker}(\ell)) = \dim(V)$$

$\rightsquigarrow$ helps for solving linear equation $\ell(x) = b$

Example: $V = W = \mathcal{P}_3(\mathbb{R})$ together with monomial basis $(m_3, m_2, m_1, m_0) =: \mathcal{B}$

with $m_0 : x \mapsto 1$, $m_k : x \mapsto x^k$

$$\begin{aligned} \ell : V &\to W \\ p &\mapsto p' \end{aligned} \quad \Longrightarrow \quad \ell(m_k) = k \cdot m_{k-1}, \quad \ell(m_0) = 0$$

matrix representation: $\ell_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\operatorname{Ker}(\ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$\operatorname{Ran}(\ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Recall general picture: $\ell = \Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}$

$$\begin{tikzcd} V \arrow[r, "\ell"] \arrow[d, "\Phi_{\mathcal{B}}"'] & W \arrow[d, "\Phi_{\mathcal{B}}"] \\ \mathbb{R}^4 \arrow[r, "\ell_{\mathcal{B} \leftarrow \mathcal{B}}"] & \mathbb{R}^4 \end{tikzcd}$$

$$\begin{aligned} \operatorname{Ker}(\ell) &= \operatorname{Ker}(\Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}) \\ &= \Phi_{\mathcal{B}}^{-1} \operatorname{Ker}(\ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \operatorname{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \operatorname{Span}(m_0) \end{aligned}$$

$$\begin{aligned} \operatorname{Ran}(\ell) &= \operatorname{Ran}(\Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B} \leftarrow \mathcal{B}} \circ \Phi_{\mathcal{B}}) \\ &= \Phi_{\mathcal{B}}^{-1} \operatorname{Ran}(\ell_{\mathcal{B} \leftarrow \mathcal{B}}) = \Phi_{\mathcal{B}}^{-1} \operatorname{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \operatorname{Span}(m_2, m_1, m_0) \end{aligned}$$

Linear equation: $\ell(p) = g$?

solutions give antiderivatives/primitives for g

$$\Longrightarrow \quad \mathcal{S} = \emptyset \quad \text{or} \quad \mathcal{S} = \tilde{p} + \operatorname{Ker}(\ell) \quad \text{with} \quad \tilde{p}' = g$$