

Abstract Linear Algebra - Part 31

$l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces (finite-dimensional).

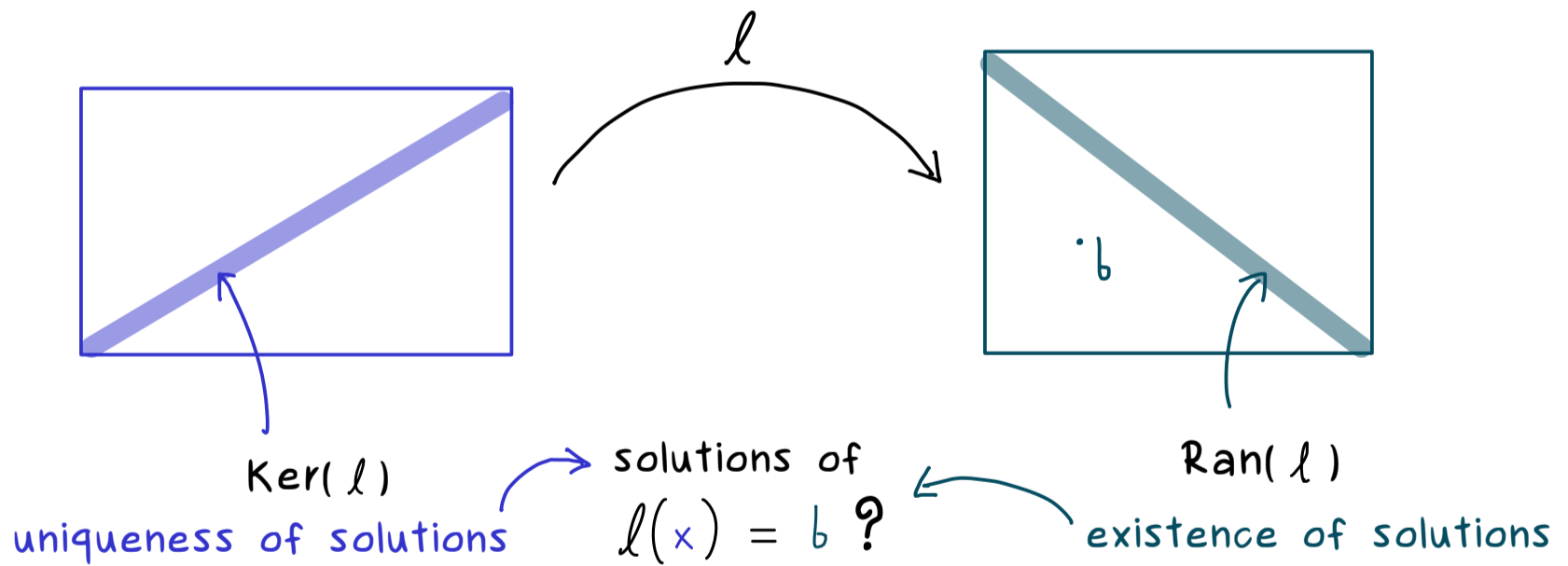
For $b \in W$:

$$l(x) = b \quad \text{solutions } x \in V$$

matrix representation

$$l_{\mathcal{B} \leftarrow \mathcal{B}} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \quad \left(\begin{array}{l} \text{system of} \\ \text{linear equations} \end{array} \right)$$

Definition: $\text{Ker}(l) := \{x \in V \mid l(x) = 0\}$ kernel of the linear map l
 $\text{Ran}(l) := \{w \in W \mid \text{there is } x \in V \text{ with } l(x) = w\}$ range of l



Proposition: $l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces, $b \in W$.

The solution set $S := \{x \in V \mid l(x) = b\}$

is either empty or an affine subspace: $S = \emptyset$ or

$$S = x_0 + \text{Ker}(l) \quad (\text{with } x_0 \in V)$$

Proof: Assume $x_0 \in S$ ($l(x_0) = b$).

Take any $v \in V$ and look at $x_0 + v$:

$$\begin{aligned} x_0 + v \in S &\iff l(x_0 + v) = b \iff \overset{\text{linear map}}{l(x_0)} + l(v) = \overset{b}{b} \\ &\iff l(v) = 0 \iff v \in \text{Ker}(l) \quad \square \end{aligned}$$

Rank-nullity theorem: $l: V \rightarrow W$ linear, V, W \mathbb{F} -vector spaces (finite-dimensional)

$$\dim(\text{Ran}(l)) + \dim(\text{Ker}(l)) = \dim(V)$$

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with matrix representations

$$\rightsquigarrow \dim(\text{Ran}(l_{\mathcal{C} \leftarrow \mathcal{B}})) + \dim(\text{Ker}(l_{\mathcal{C} \leftarrow \mathcal{B}})) = n$$