Abstract Linear Algebra - Part 31

$\ell : V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces (finite-dimensional).

For $b \in W$: $\ell(x) = b$ solutions $x \in V$

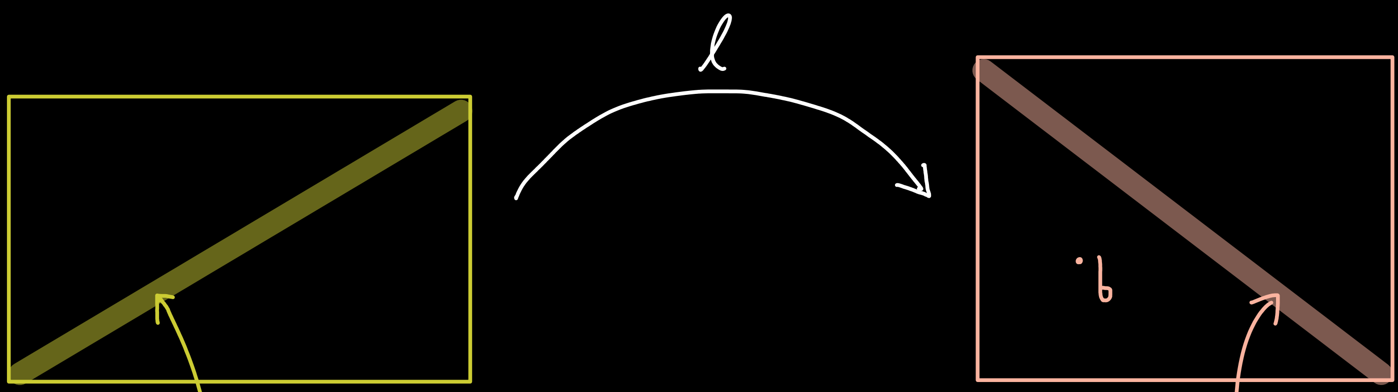
matrix representation

$\ell_{\mathcal{C} \leftarrow \mathcal{B}} \begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$ (system of linear equations)

<u>Definition:</u>

$\mathrm{Ker}(\ell) := \{ x \in V \mid \ell(x) = 0 \}$ <u>kernel</u> of the linear map ℓ

$\mathrm{Ran}(\ell) := \{ w \in W \mid \text{there is } x \in V \text{ with } \ell(x) = w \}$ <u>range</u> of ℓ



Ker(ℓ) uniqueness of solutions

solutions of $\ell(x) = b$?

Ran(ℓ) existence of solutions

Proposition: $\ell: V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces, $b \in W$.

The solution set $\mathcal{S} := \{ x \in V \mid \ell(x) = b \}$

is either empty or an affine subspace: $\mathcal{S} = \emptyset$ or

$$\mathcal{S} = x_0 + \mathrm{Ker}(\ell) \quad (\text{with } x_0 \in V)$$

Proof: Assume $x_0 \in \mathcal{S}$ $\big(\ell(x_0) = b\big)$.

Take any $v \in V$ and look at $x_0 + v$:

$$x_0 + v \in \mathcal{S} \iff \ell(x_0 + v) = b \overset{\text{linear map}}{\iff} \underbrace{\ell(x_0)}_{=b} + \ell(v) = b$$

$$\iff \ell(v) = 0 \iff v \in \mathrm{Ker}(\ell) \qquad \square$$

Rank-nullity theorem: $\ell: V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces (finite-dimensional)

$$\dim\big(\mathrm{Ran}(\ell)\big) + \dim\big(\mathrm{Ker}(\ell)\big) = \dim(V)$$

$\|$ part 28/29 $\qquad \|$ $\qquad \|$

with matrix representations

$$\rightsquigarrow \quad \dim\big(\mathrm{Ran}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})\big) + \dim\big(\mathrm{Ker}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})\big) = n$$