Abstract Linear Algebra - Part 29

Equivalence relation: $A, B \in \mathbb{F}^{m \times n}$, $A \sim B$

they both represent the same linear map $\ell : V \longrightarrow W$

there are invertible matrices S, T with $B = S\,A\,T$.

kernel and range?

$$\mathrm{Ker}(B) = \mathrm{Ker}(SAT) = \{ x \in \mathbb{F}^n \mid A\underbrace{Tx}_{\in \mathrm{Ker}(A)} = 0 \} = T^{-1}\,\mathrm{Ker}(A)$$

$$\mathrm{Ran}(B) = \mathrm{Ran}(SAT) = \{ SATx \mid x \in \mathbb{F}^n \}$$

$$= \{ S\underbrace{A\tilde{x}}_{\in \mathrm{Ran}(A)} \mid \tilde{x} \in \mathbb{F}^n \} = S\,\mathrm{Ran}(A)$$

Result: $A \sim B \implies$

$$\begin{array}{ccc} \mathrm{rank}(A) & = & \mathrm{rank}(B) \\ + & & + \\ \mathrm{nullity}(A) & = & \mathrm{nullity}(B) \\ \| & & \| \\ n & & n \end{array}$$

Proposition: For $A, B \in \mathbb{F}^{m \times n}$, we have:

$$A \sim B \iff \mathrm{rank}(A) = \mathrm{rank}(B)$$

Proof:

$$A \sim \begin{pmatrix} \blacksquare & & & & \\ & & \blacksquare & & \\ & & & \blacksquare & \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & 0 & \cdots \\ & & \boxed{1} & 0 & \cdots \\ & & & \boxed{1} & 0 & \cdots \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{1} & \end{pmatrix}$$

Gaussian elimination

row echelon form

backwards substitution

column/row exchanges

$$\Rightarrow \quad A \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad r = \operatorname{rank}(A)$$

$$B \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} \qquad r = \operatorname{rank}(B)$$

$\square$