

Abstract Linear Algebra - Part 29

Equivalence relation: $A, B \in \mathbb{F}^{m \times n}$, $A \sim B$ \leftarrow they both represent the same linear map $\ell: V \rightarrow W$
 \leftarrow there are invertible matrices S, T with $B = S A T$.

kernel and range?

$$\text{Ker}(B) = \text{Ker}(SAT) = \left\{ x \in \mathbb{F}^n \mid \underbrace{ATx}_{\in \text{Ker}(A)} = 0 \right\} = T^{-1} \text{Ker}(A)$$

$$\begin{aligned} \text{Ran}(B) &= \text{Ran}(SAT) = \left\{ SATx \mid x \in \mathbb{F}^n \right\} \\ &= \left\{ SA \underbrace{\tilde{x}}_{\in \text{Ran}(A)} \mid \tilde{x} \in \mathbb{F}^n \right\} = S \text{Ran}(A) \end{aligned}$$

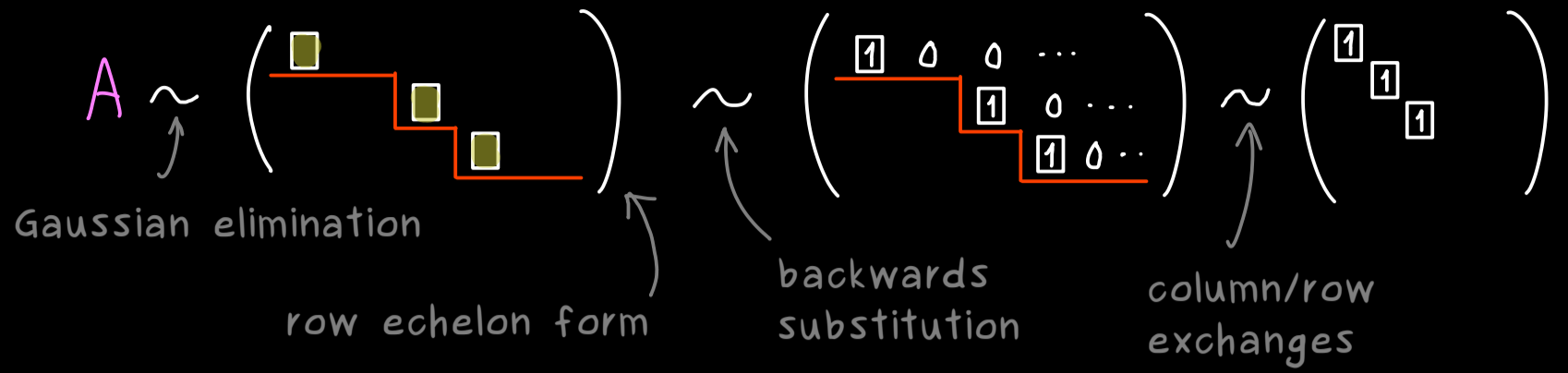
Result: $A \sim B \implies$

$$\begin{array}{ccc} \text{rank}(A) & = & \text{rank}(B) \\ + & & + \\ \text{nullity}(A) & = & \text{nullity}(B) \\ \parallel & & \parallel \\ n & & n \end{array}$$

Proposition: For $A, B \in \mathbb{F}^{m \times n}$, we have:

$$A \sim B \iff \text{rank}(A) = \text{rank}(B)$$

Proof:



$$\Rightarrow A \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}$$
$$B \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix}$$

with $r = \text{rank}(A)$

$r = \text{rank}(B)$

□