Abstract Linear Algebra - Part 27

$$\begin{tikzcd}[column sep=huge, row sep=large]
V \text{ (basis } \mathcal{B}\text{, basis } \widetilde{\mathcal{B}}\text{)} \arrow[rr, "\ell"', "\text{linear}"] \arrow[d, shift left, "\Phi_{\mathcal{B}}"] \arrow[dd, bend right=60, "\Phi_{\widetilde{\mathcal{B}}}"'] & & W \text{ (basis } \mathcal{C}\text{, basis } \widetilde{\mathcal{C}}\text{)} \arrow[d, shift left, "\Phi_{\mathcal{C}}"] \arrow[dd, bend left=60, "\Phi_{\widetilde{\mathcal{C}}}"] \\
\mathbb{F}^n \text{ (standard basis)} \arrow[u, shift left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[rr, "\text{matrix}", "\ell_{\mathcal{C}\leftarrow\mathcal{B}}"'] \arrow[d, shift left, "T_{\widetilde{\mathcal{B}}\leftarrow\mathcal{B}}"] \arrow[d, phantom, "\text{change of basis}"] & & \mathbb{F}^m \text{ (standard basis)} \arrow[u, shift left, "\Phi_{\mathcal{C}}^{-1}"] \arrow[d, shift left, "T_{\widetilde{\mathcal{C}}\leftarrow\mathcal{C}}"] \arrow[d, phantom, "\text{change of basis}"] \\
\mathbb{F}^n \text{ (standard basis)} \arrow[u, shift left, "T_{\mathcal{B}\leftarrow\widetilde{\mathcal{B}}}"] \arrow[uu, bend left=80, "\Phi_{\widetilde{\mathcal{B}}}^{-1}"] \arrow[rr, "\text{matrix}", "\ell_{\widetilde{\mathcal{C}}\leftarrow\widetilde{\mathcal{B}}}"'] & & \mathbb{F}^m \text{ (standard basis)} \arrow[u, shift left, "T_{\mathcal{C}\leftarrow\widetilde{\mathcal{C}}}"] \arrow[uu, bend right=80, "\Phi_{\widetilde{\mathcal{C}}}^{-1}"']
\end{tikzcd}$$

Result:

$$\boxed{\ell_{\widetilde{\mathcal{C}}\leftarrow\widetilde{\mathcal{B}}} = T_{\widetilde{\mathcal{C}}\leftarrow\mathcal{C}} \, \ell_{\mathcal{C}\leftarrow\mathcal{B}} \, T_{\mathcal{B}\leftarrow\widetilde{\mathcal{B}}}}$$

Example: $\ell : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\ell(p) = p'$ linear map!

$$\mathcal{B} = (m_3, m_2, m_1, m_0) \qquad \mathcal{C} = (m_2, m_1, m_0)$$

$$\widetilde{\mathcal{B}} = (2m_3 - m_1,\ m_2 + m_0,\ m_1 + m_0,\ m_1 - m_0) , \quad \widetilde{\mathcal{C}} = (m_2 - \tfrac{1}{2}m_1,\ m_2 + \tfrac{1}{2}m_1,\ m_0)$$

matrix representation: $\ell_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

change-of-basis matrices: $T_{\mathcal{B}\leftarrow\widetilde{\mathcal{B}}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

$$T_{\mathcal{C}\leftarrow\tilde{\mathcal{C}}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{inverse}} T_{\tilde{\mathcal{C}}\leftarrow\mathcal{C}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\ell_{\tilde{\mathcal{C}}\leftarrow\hat{\mathcal{B}}} = T_{\tilde{\mathcal{C}}\leftarrow\mathcal{C}}\, \ell_{\mathcal{C}\leftarrow\mathcal{B}}\, T_{\mathcal{B}\leftarrow\hat{\mathcal{B}}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$