Abstract Linear Algebra – Part 26

Matrix representations:

$$
\begin{tikzcd}[column sep=large]
\begin{array}{c} V \\ \text{with basis:} \\ \mathcal{B} = (b_1, \dots, b_n) \end{array} \arrow[r, bend left, "\ell" , "\text{linear}"'] \arrow[rr, bend left=40, "k \circ \ell"] \arrow[d, bend left, "\Phi_{\mathcal{B}}"] & \begin{array}{c} W \\ \text{with basis:} \\ \mathcal{C} = (c_1, \dots, c_m) \end{array} \arrow[r, bend left, "k", "\text{linear}"'] \arrow[d, bend left, "\Phi_{\mathcal{C}}"] & \begin{array}{c} U \\ \text{with basis:} \\ \mathcal{D} = (d_1, \dots, d_k) \end{array} \arrow[d, bend left, "\Phi_{\mathcal{D}}"] \\
\boxed{\begin{array}{c} \mathbb{F}^n \\ \text{(standard basis)} \end{array}} \arrow[u, bend left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[r, bend left, "\ell_{\mathcal{C} \leftarrow \mathcal{B}}", "\in \mathbb{F}^{m \times n}"'] & \boxed{\begin{array}{c} \mathbb{F}^m \\ \text{(standard basis)} \end{array}} \arrow[u, bend left, "\Phi_{\mathcal{C}}^{-1}"] \arrow[r, bend left, "k_{\mathcal{D} \leftarrow \mathcal{C}}", "\in \mathbb{F}^{k \times m}"'] & \boxed{\begin{array}{c} \mathbb{F}^k \\ \text{(standard basis)} \end{array}} \arrow[u, bend left, "\Phi_{\mathcal{D}}^{-1}"]
\end{tikzcd}
$$

We get: $(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}}$ (matrix product)

Example: $\ell : \mathbb{R}^3 \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto (v_1 + v_2 + v_3) \cdot m_0 + (v_1 + v_2) \cdot m_1 + v_1 \cdot m_2$

with $m_0 : x \mapsto 1$, $m_k : x \mapsto x^k$

$k : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$, $p \mapsto \begin{pmatrix} p'(1) \\ p(1) - p''(1) \end{pmatrix}$

Picture:

$$
\begin{tikzcd}
\begin{array}{c} \mathbb{R}^3 \\ \mathcal{B} \end{array} \arrow[r, bend left, "\ell"] & \begin{array}{c} \mathcal{P}_2 \\ \mathcal{C} \end{array} \arrow[r, bend left, "k"] & \begin{array}{c} \mathbb{R}^2 \\ \mathcal{D} \end{array} \\
\text{standard basis} & (m_0, m_1, m_2) & \text{standard basis}
\end{tikzcd}
$$

$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = ?$

$$
\ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | & | \\ \Phi_{\mathcal{C}}(\ell(b_1)) & \Phi_{\mathcal{C}}(\ell(b_2)) & \Phi_{\mathcal{C}}(\ell(b_3)) \\ | & | & | \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$
k_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}
$$

$$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Corollary: $$(\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = \left(\ell_{\mathcal{C} \leftarrow \mathcal{B}}\right)^{-1}$$

$n = m$