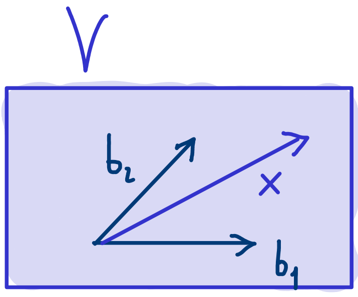
Abstract Linear Algebra – Part 25

$\ell : V \longrightarrow W$ linear: $\quad \ell(x+y) = \ell(x) + \ell(y)$

$$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$$



$\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V

$\Longrightarrow$ $x \in V$ can be written as $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$

Hence: $\quad \ell(x) = \ell(\alpha_1 b_1 + \cdots + \alpha_n b_n)$

$$= \alpha_1 \ell(b_1) + \alpha_2 \ell(b_2) + \cdots + \alpha_n \ell(b_n)$$

If know these, we know ℓ

Example: $\quad V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$, $\quad \ell : V \to W$, $\; p \mapsto p'$

is a linear map!

basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0 : x \mapsto 1$, $\quad m_k : x \mapsto x^k$

$$\ell(m_0) = 0 \quad \text{zero vector: } x \mapsto 0$$

$$\ell(m_k) = k \cdot m_{k-1} \quad , \quad k \in \{1,2,3\}$$

Result:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c} V \\ \text{with basis:} \\ \mathcal{B} = (b_1,\dots,b_n) \end{array}} \arrow[rr, bend left, "\ell \atop \text{linear}"] \arrow[d, bend left, "\Phi_{\mathcal{B}}"] & & \boxed{\begin{array}{c} W \\ \text{with basis:} \\ \mathcal{C} = (c_1,\dots,c_m) \end{array}} \arrow[d, "\Phi_{\mathcal{C}}"] \\
\boxed{\begin{array}{c} \mathbb{F}^n \\ \text{(standard basis)} \end{array}} \arrow[u, bend left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[rr, bend left, "f = \Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}"] & & \boxed{\begin{array}{c} \mathbb{F}^m \\ \text{(standard basis)} \end{array}}
\end{tikzcd}$$

linear

$f: \mathbb{F}^n \to \mathbb{F}^m$

can be represented by a matrix

$f(x) = Ax$

$\in \mathbb{F}^{m \times n}$

First column of A: $\quad f(e_1) = (\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1})(e_1) = \Phi_{\mathcal{C}}(\ell(b_1))$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Matrix representation: For a linear map $\ell: V \to W$,

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} := \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(\ell(b_1)) & \Phi_{\mathcal{C}}(\ell(b_2)) & \cdots & \Phi_{\mathcal{C}}(\ell(b_n)) \\ | & | & & | \end{pmatrix} \in \mathbb{F}^{m \times n}$$

is called the matrix representation of ℓ with respect to $\mathcal{B}$ and $\mathcal{C}$.

Example (from before)

$V = \mathcal{P}_3(\mathbb{R})$ basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$\ell: V \to W$
$p \mapsto p'$

is a linear map!

$W = \mathcal{P}_2(\mathbb{R})$ basis: $\mathcal{C} = (c_1, c_2, c_3) = (m_0, m_1, m_2)$

$$\Phi_{\mathcal{C}}(\ell(b_1)) = \Phi_{\mathcal{C}}(\ell(m_0)) = \Phi_{\mathcal{C}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$$

$$\Phi_{\mathcal{C}}(\ell(b_2)) = \Phi_{\mathcal{C}}(\ell(m_1)) = \Phi_{\mathcal{C}}(m_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$$

$$\vdots$$

$$\Rightarrow \quad \ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \text{matrix representation of } \ell$$