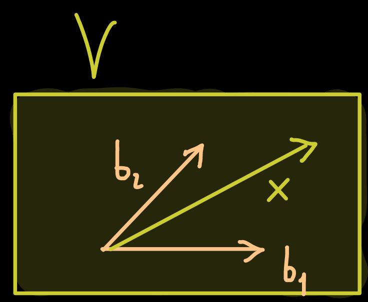
Abstract Linear Algebra - Part 25

$\ell: V \longrightarrow W$ linear: $\ell(x+y) = \ell(x) + \ell(y)$

$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$



$\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V

$\Longrightarrow$ $x \in V$ can be written as $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$

Hence: $\ell(x) = \ell(\alpha_1 b_1 + \cdots + \alpha_n b_n)$

$= \alpha_1 \ell(b_1) + \alpha_2 \ell(b_2) + \cdots + \alpha_n \ell(b_n)$

If know these, we know ℓ

Example: $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$, $\ell: V \to W$
$p \mapsto p'$

is a linear map!

basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$\ell(m_0) = 0$ zero vector: $x \mapsto 0$

$\ell(m_k) = k \cdot m_{k-1}$, $k \in \{1,2,3\}$

Result:

$$
\begin{tikzcd}[column sep=huge, row sep=large]
V \text{ with basis: } \mathcal{B}=(b_1,\dots,b_n) \arrow[r, "\ell", "\text{linear}"'] \arrow[d, bend left, "\Phi_{\mathcal{B}}"] & W \text{ with basis: } \mathcal{C}=(c_1,\dots,c_m) \arrow[d, "\Phi_{\mathcal{C}}"] \\
\boxed{\mathbb{F}^n \text{ (standard basis)}} \arrow[u, bend left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[r, "f = \Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}", "\text{linear}"'] & \boxed{\mathbb{F}^m \text{ (standard basis)}}
\end{tikzcd}
$$

linear
$f: \mathbb{F}^n \to \mathbb{F}^m$

can be represented by a matrix

$$f(x) = A x$$

$\in \mathbb{F}^{m\times n}$

First column of A: $f(e_1) = \left(\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}\right)(e_1) = \Phi_{\mathcal{C}}(\ell(b_1))$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Matrix representation: For a linear map $\ell: V \to W$,

$$\ell_{\mathcal{C}\leftarrow\mathcal{B}} := \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(\ell(b_1)) & \Phi_{\mathcal{C}}(\ell(b_2)) & \cdots & \Phi_{\mathcal{C}}(\ell(b_n)) \\ | & | & & | \end{pmatrix} \in \mathbb{F}^{m\times n}$$

is called the matrix representation of ℓ with respect to $\mathcal{B}$ and $\mathcal{C}$.

Example (from before)

$V = \mathcal{P}_3(\mathbb{R})$ basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$\ell: V \to W$
$p \mapsto p'$

is a linear map!

$W = \mathcal{P}_2(\mathbb{R})$ basis: $\mathcal{C} = (c_1, c_2, c_3) = (m_0, m_1, m_2)$

$$\Phi_{\mathcal{C}}(\ell(b_1)) = \Phi_{\mathcal{C}}(\ell(m_0)) = \Phi_{\mathcal{C}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$$

$$\Phi_{\mathcal{C}}(\ell(b_2)) = \Phi_{\mathcal{C}}(\ell(m_1)) = \Phi_{\mathcal{C}}(m_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$$

$\vdots$

$$\Rightarrow \ell_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ matrix representation of ℓ