Abstract Linear Algebra – Part 23

Recall: linear map or linear operator $\ell: V \longrightarrow W$:

$$\ell(x+y) = \ell(x) + \ell(y)$$
$$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$$

Definition: Let V, W be two $\mathbb{F}$-vector spaces. (same $\mathbb{F}$ for both)

For $k: V \longrightarrow W$, $\ell: V \longrightarrow W$ linear maps, we define:

$$k+\ell: V \longrightarrow W, \quad (k+\ell)(x) := k(x) + \ell(x)$$

(given $\lambda \in \mathbb{F}$) $$\lambda \cdot \ell: V \longrightarrow W, \quad (\lambda \cdot \ell)(x) := \lambda \cdot \ell(x)$$

$$\begin{tikzcd} \boxed{V} \arrow[r, bend left, "\ell"] & \boxed{W} \\ \boxed{V} \arrow[r, bend left, "\lambda\cdot\ell"] & \boxed{W} \end{tikzcd}$$

Result: With $+, \cdot$ from above, the set $\mathcal{L}(V,W) = \{ \ell: V \longrightarrow W \mid \text{linear} \}$ forms an $\mathbb{F}$-vector space.

Zero vector $o \in \mathcal{L}(V,W)$ is given by the zero map $o(x) = o_W$ for all $x \in V$.

zero vector in W

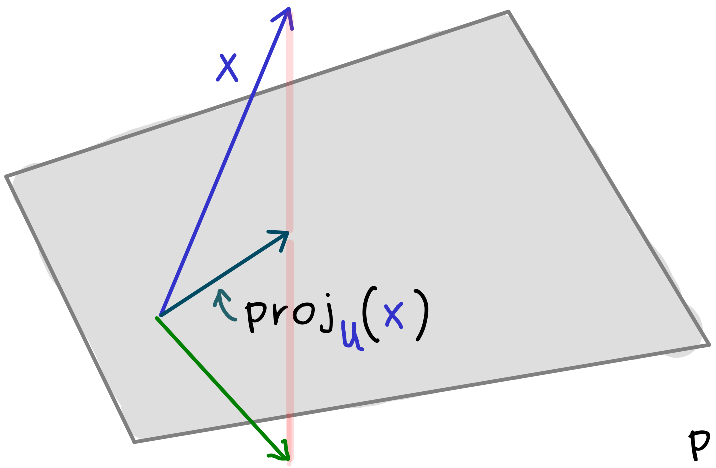
Example: V with inner product $\langle \cdot , \cdot \rangle$ and ONB $(e_1, e_2, \dots, e_n)$.

$U = \mathrm{Span}(e_1, e_2, \dots, e_{n-1})$

Orthogonal projection onto U : $\mathrm{proj}_U : V \longrightarrow V$

$$x \longmapsto \sum_{j=1}^{n-1} e_j \langle e_j , x \rangle$$

linear map



$\mathrm{proj}_{U^\perp} : V \longrightarrow V$

$$x \longmapsto e_n \langle e_n , x \rangle$$

linear map

Addition: $\mathrm{proj}_U + \mathrm{proj}_{U^\perp} = \mathrm{id}_V$

Subtraction: $\mathrm{proj}_U - \mathrm{proj}_{U^\perp} = \mathrm{id}_V - 2 \cdot \mathrm{proj}_{U^\perp}$ reflection

Composition:

$$\begin{tikzcd} \boxed{U} \arrow[r, bend left, "k", "\text{linear}"'] \arrow[rr, bend right, "\ell \circ k \ \text{linear}"'] & \boxed{V} \arrow[r, bend left, "\ell", "\text{linear}"'] & \boxed{W} \end{tikzcd}$$

$$k \in \mathcal{L}(U, V) \, , \ \ell \in \mathcal{L}(V, W) \implies \ell \circ k \in \mathcal{L}(U, W)$$

Example: $\mathrm{proj}_U \circ \mathrm{proj}_U = \mathrm{proj}_U$, $\mathrm{proj}_U \circ \mathrm{proj}_{U^\perp} = 0$

zero vector in $\mathcal{L}(V, V)$