



Abstract Linear Algebra - Part 23

Recall: linear map or linear operator $l: V \rightarrow W$:

$$l(x+y) = l(x) + l(y)$$

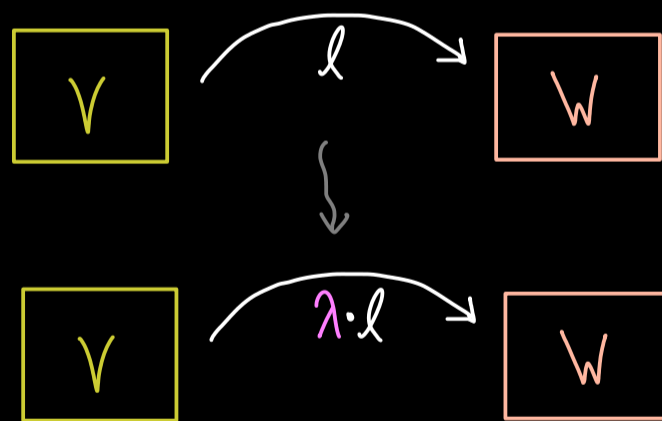
$$l(\lambda \cdot x) = \lambda \cdot l(x)$$

Definition: Let V, W be two \mathbb{F} -vector spaces. (same \mathbb{F} for both)

For $k: V \rightarrow W$, $l: V \rightarrow W$ linear maps, we define:

$$k+l: V \rightarrow W, \quad (k+l)(x) := k(x) + l(x)$$

$$\text{(given } \lambda \in \mathbb{F} \text{)} \quad \lambda \cdot l: V \rightarrow W, \quad (\lambda \cdot l)(x) := \lambda \cdot l(x)$$



Result: With $+$, \cdot from above, the set $\mathcal{L}(V, W) = \{ l: V \rightarrow W \mid \text{linear} \}$

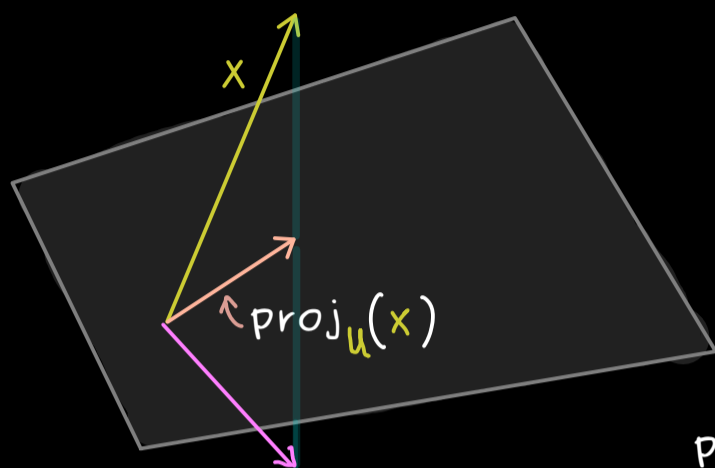
forms an \mathbb{F} -vector space.

Zero vector $0 \in \mathcal{L}(V, W)$ is given by the zero map $0(x) = 0_W$ for all $x \in V$.
↑ zero vector in W

Example: V with inner product $\langle \cdot, \cdot \rangle$ and ONB (e_1, e_2, \dots, e_n) .

$$U = \text{span}(e_1, e_2, \dots, e_{n-1})$$

Orthogonal projection onto U : $\text{proj}_U: V \rightarrow V$



$$x \mapsto \sum_{j=1}^{n-1} e_j \langle e_j, x \rangle$$

linear map

$\text{proj}_{U^\perp}: V \rightarrow V$

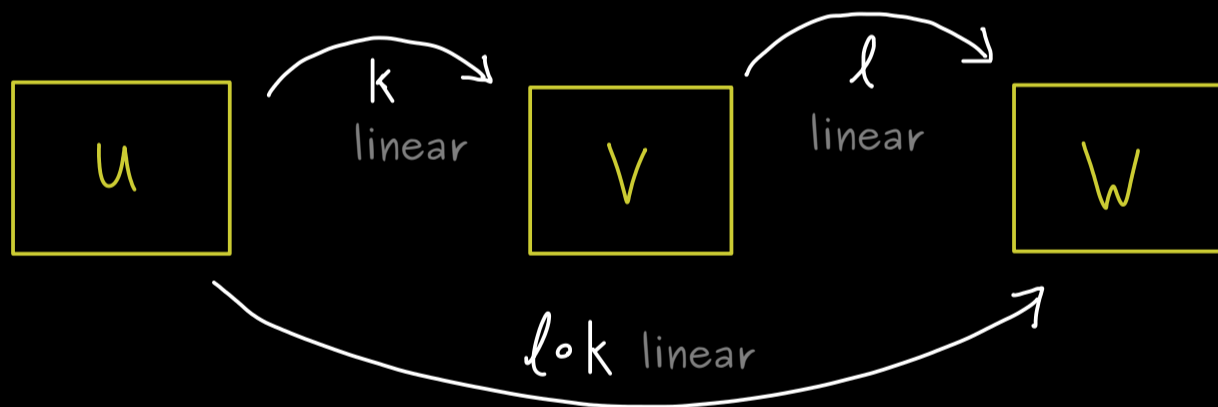
$$x \mapsto e_n \langle e_n, x \rangle$$

linear map

Addition: $\text{proj}_U + \text{proj}_{U^\perp} = \text{id}_V$

Subtraction: $\text{proj}_U - \text{proj}_{U^\perp} = \text{id}_V - 2 \cdot \text{proj}_{U^\perp}$ reflection

Composition:



$$k \in \mathcal{L}(U, V), l \in \mathcal{L}(V, W) \Rightarrow l \circ k \in \mathcal{L}(U, W)$$

Example: $\text{proj}_U \circ \text{proj}_U = \text{proj}_U$, $\text{proj}_U \circ \text{proj}_{U^\perp} = 0$

zero vector in $\mathcal{L}(V, V)$