Abstract Linear Algebra – Part 23

Recall: linear map or linear operator $\ell : V \longrightarrow W$:

$$\ell(x+y) = \ell(x) + \ell(y)$$
$$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$$

Definition: Let V, W be two $\mathbb{F}$–vector spaces. (same $\mathbb{F}$ for both)

For $k : V \longrightarrow W$, $\ell : V \longrightarrow W$ linear maps, we define:

$$k + \ell : V \longrightarrow W \,, \quad (k+\ell)(x) := k(x) + \ell(x)$$
(given $\lambda \in \mathbb{F}$) $$\lambda \cdot \ell : V \longrightarrow W \,, \quad (\lambda \cdot \ell)(x) := \lambda \cdot \ell(x)$$

$$\begin{tikzcd} \boxed{V} \arrow[r, bend left, "\ell"] & \boxed{W} \end{tikzcd}$$
$$\begin{tikzcd} \boxed{V} \arrow[r, bend left, "\lambda \cdot \ell"] & \boxed{W} \end{tikzcd}$$

Result: With $+, \cdot$ from above, the set $\mathcal{L}(V, W) = \{ \ell : V \longrightarrow W \mid \text{linear} \}$ forms an $\mathbb{F}$–vector space.

Zero vector $o \in \mathcal{L}(V,W)$ is given by the zero map $o(x) = o_W$ for all $x \in V$.

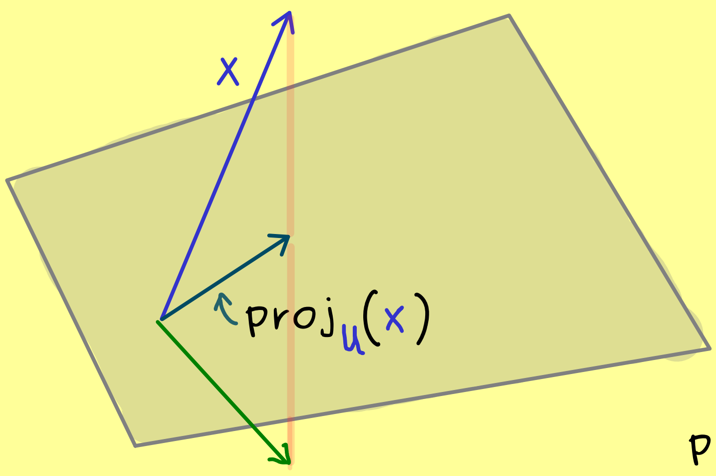
zero vector in W

Example: V with inner product $\langle \cdot , \cdot \rangle$ and ONB $(e_1, e_2, \dots, e_n)$.

$U = \text{Span}(e_1, e_2, \dots, e_{n-1})$

Orthogonal projection onto U : $\text{proj}_U : V \longrightarrow V$ linear map

$$x \longmapsto \sum_{j=1}^{n-1} e_j \langle e_j , x \rangle$$



$\text{proj}_{U^\perp} : V \longrightarrow V$ linear map

$$x \longmapsto e_n \langle e_n , x \rangle$$

Addition: $\text{proj}_U + \text{proj}_{U^\perp} = \text{id}_V$

Subtraction: $\text{proj}_U - \text{proj}_{U^\perp} = \text{id}_V - 2 \cdot \text{proj}_{U^\perp}$ reflection

Composition:

$$\begin{tikzcd} \boxed{U} \arrow[r, bend left, "k \text{ linear}"] \arrow[rr, bend right, "\ell \circ k \text{ linear}"'] & \boxed{V} \arrow[r, bend left, "\ell \text{ linear}"] & \boxed{W} \end{tikzcd}$$

$$k \in \mathcal{L}(U, V) \,, \; \ell \in \mathcal{L}(V, W) \implies \ell \circ k \in \mathcal{L}(U, W)$$

Example: $\text{proj}_U \circ \text{proj}_U = \text{proj}_U$, $\text{proj}_U \circ \text{proj}_{U^\perp} = o$

zero vector in $\mathcal{L}(V,V)$