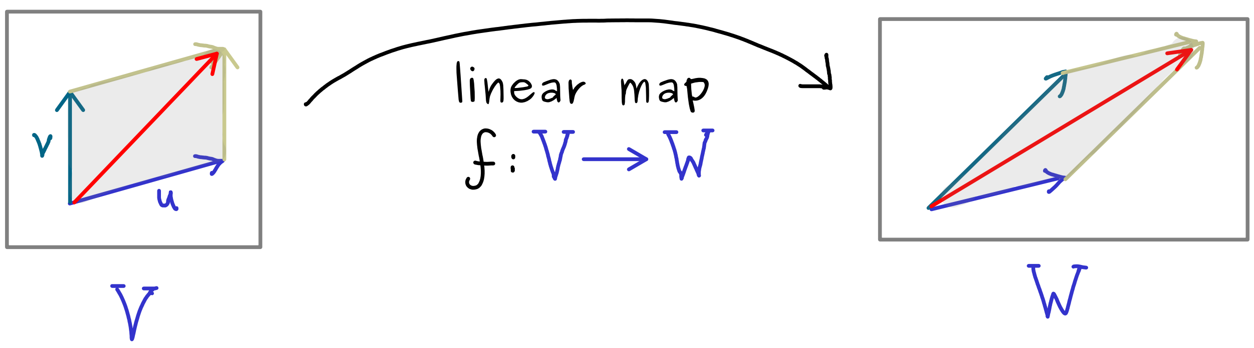
Abstract Linear Algebra – Part 22



Recall: $f : \mathbb{R}^n \to \mathbb{R}^m$ linear $\Longleftrightarrow$ matrix $A \in \mathbb{R}^{m \times n}$

Definition: Let V, W be two $\mathbb{F}$-vector spaces. (same $\mathbb{F}$ for both)

A map $f : V \to W$ is called linear if:

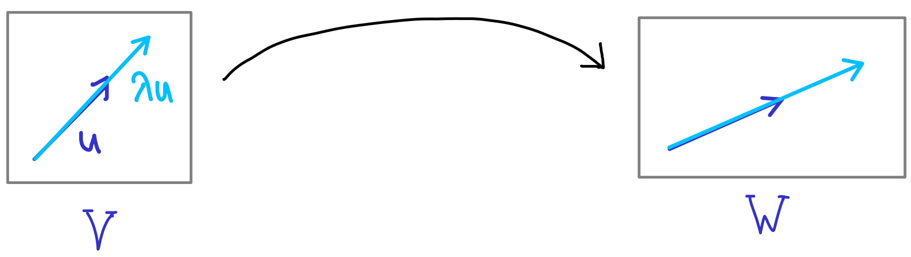
(1) $f(u + v) = f(u) + f(v)$

vector addition in V — vector addition in W

(2) $f(\lambda \cdot u) = \lambda \cdot f(u)$

scalar multiplication in V — scalar multiplication in W

for all $u, v \in V$, $\lambda \in \mathbb{F}$.



Remember: $f(0_V) = f(0 \cdot u) \overset{(2)}{=} 0 \cdot f(u) = 0_W$

Example: (a) $V = \mathbb{F}^3$, $W = \mathbb{F}$, $a \in V$.

$f(u) := \langle a, u \rangle_{\text{standard}}$ is a linear map.

$\qquad = a^* u$ (matrix multiplication)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^* = \begin{pmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} \end{pmatrix}$$

(transpose + complex conjugation)

(b) $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$

$$\begin{aligned} \ell : V &\to W \\ p &\mapsto p' \end{aligned}$$

is a linear map!

$$\ell(x \mapsto x^2) = x \mapsto 2x$$

$$\ell(p+q) = (p+q)' = p' + q' = \ell(p) + \ell(q)$$

$$\ell(\lambda p) = (\lambda p)' = \lambda p' = \lambda \ell(p)$$