Abstract Linear Algebra – Part 21

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

basis of U: $(u_1, u_2, \dots, u_k)$ $\rightsquigarrow$ ONB of U: $(b_1, b_2, \dots, b_k)$

Gram-Schmidt process/algorithm

Example: $V = \mathcal{P}([-1,1], \mathbb{R})$ polynomial space with inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x) \, dx$$

Take $U = \mathcal{P}_2([-1,1], \mathbb{R})$ with basis (m_0, m_1, m_2)

(polynomials of degree ≤ 2)

not ONB!

$m_0 : x \mapsto 1$

$m_1 : x \mapsto x$

$m_2 : x \mapsto x^2$

Gram-Schmidt orthonormalization:

(1) Normalize first vector: $\|m_0\|^2 = \langle m_0, m_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2$

$$b_0 := \frac{1}{\|m_0\|} \cdot m_0 = \frac{1}{\sqrt{2}} m_0 , \qquad b_0(x) = \frac{1}{\sqrt{2}}$$

(2) Next vector m_1:

normal component: $\widetilde{b}_1 = m_1 - b_0 \underbrace{\langle b_0, m_1 \rangle}_{= \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot x \, dx = 0} = m_1$

normalize it: $b_1 := \frac{1}{\|\widetilde{b}_1\|} \widetilde{b}_1$, $\|\widetilde{b}_1\|^2 = \int_{-1}^{1} x \cdot x \, dx = \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{2}{3}$

$$= \sqrt{\tfrac{3}{2}}\, m_1 , \qquad b_1(x) = \sqrt{\tfrac{3}{2}}\, x$$

(3) Next vector m_2:

normal component:

$$\tilde{b}_2 = m_2 - b_0 \underbrace{\langle b_0, m_2 \rangle}_{= \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot x^2 \, dx \, = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3}} - b_1 \underbrace{\langle b_1, m_2 \rangle}_{= \int_{-1}^{1} \sqrt{\frac{3}{2}} x \cdot x^2 \, dx \, = 0}$$

$$= m_2 - \frac{1}{3} m_0 , \qquad \tilde{b}_2(x) = x^2 - \frac{1}{3}$$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$,

$$\|\tilde{b}_2\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right) dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9}\right) dx$$

$$= \frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \Big|_{-1}^{1} = \frac{8}{45}$$

$$\Longrightarrow \quad b_2(x) = \sqrt{\frac{45}{8}} \cdot \left(x^2 - \frac{1}{3}\right)$$

$\rightsquigarrow$ ONB for $\mathcal{P}_2([-1,1], \mathbb{R})$

(Legendre polynomials)