

Abstract Linear Algebra - Part 20

V \mathbb{F} -vector space, inner product $\langle \cdot, \cdot \rangle$, $U \subseteq V$ k -dimensional subspace.

basis of U : (u_1, u_2, \dots, u_k) \rightsquigarrow ONB of U : (b_1, b_2, \dots, b_k)
Gram-Schmidt process/algorithm $\langle b_i, b_j \rangle = \delta_{ij}$

Gram-Schmidt orthonormalization:

(1) Normalize first vector:

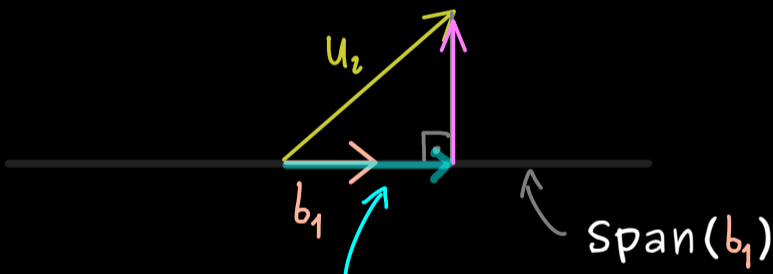
$$b_1 := \frac{1}{\|u_1\|} \cdot u_1$$

where

$$\|u_1\| := \sqrt{\langle u_1, u_1 \rangle}$$

length = 1?

(2) Next vector u_2 :



orthogonal projection of u_2 onto $\text{Span}(b_1)$:

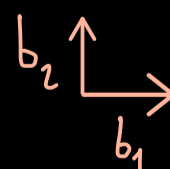
$$\hookrightarrow u_2|_{\text{Span}(b_1)} = b_1 \langle b_1, u_2 \rangle$$

normal component:

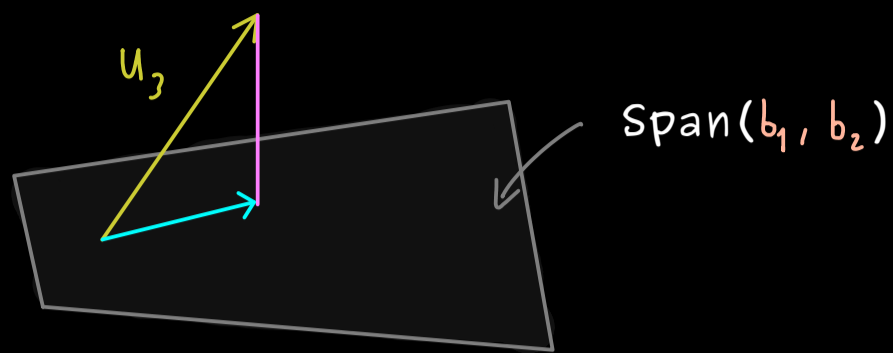
$$\tilde{b}_2 = u_2 - b_1 \langle b_1, u_2 \rangle$$

normalize it:

$$b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$$



(3) Next vector u_3 :



orthogonal projection of u_3 onto $\text{Span}(b_1, b_2)$:

$$\hookrightarrow u_3|_{\text{Span}(b_1, b_2)} := b_1 \langle b_1, u_3 \rangle + b_2 \langle b_2, u_3 \rangle$$

normal component: $\tilde{b}_3 = u_3 - b_1 \langle b_1, u_3 \rangle - b_2 \langle b_2, u_3 \rangle$

normalize it: $b_3 := \frac{1}{\|\tilde{b}_3\|} \tilde{b}_3$

•
• continue!
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(k) Next vector u_k :



orthogonal projection of u_k onto $\text{Span}(b_1, b_2, \dots, b_{k-1})$

$$\hookrightarrow u_k|_{\text{Span}(b_1, b_2, \dots, b_{k-1})} := \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$$

normal component: $\tilde{b}_k = u_k - \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$

normalize it: $b_k := \frac{1}{\|\tilde{b}_k\|} \tilde{b}_k \implies \text{ONB of } U : (b_1, b_2, \dots, b_k)$