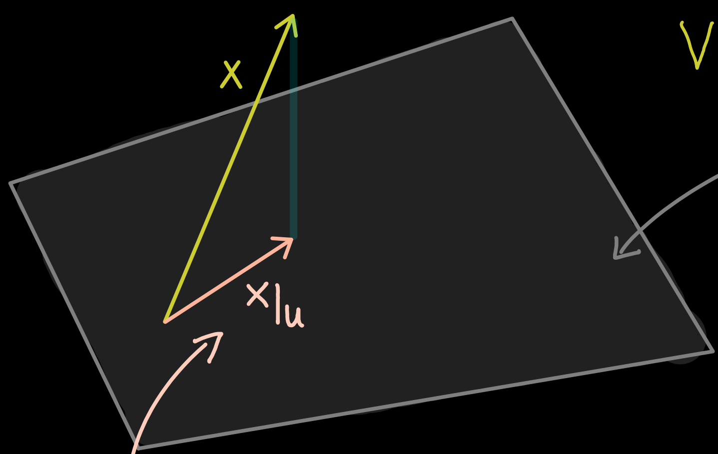
Abstract Linear Algebra - Part 19



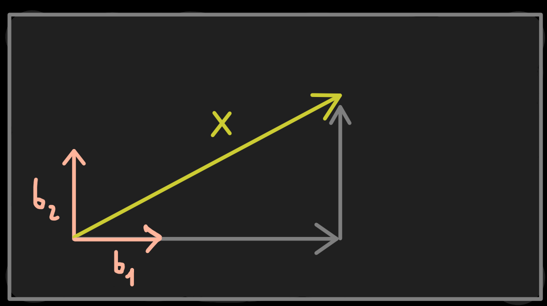
V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,

$U \subseteq V$ k-dimensional subspace,

$\mathcal{B} = (b_1, b_2, \dots, b_k)$ ONB of U.

orthogonal projection: $x|_U = \sum_{j=1}^{k} b_j \underbrace{\langle b_j , x \rangle}_{\text{scalars}}$

The case $x \in U$:



$x = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_k b_k$

How to find?

↳ easy for ONB!

Result: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ONB of U.

Then for each $u \in U$ we have the linear combination

$$u = \sum_{j=1}^{k} b_j \underbrace{\langle b_j , u \rangle}_{\in \mathbb{F}}$$

(Fourier expansion of u w.r.t. $\mathcal{B}$)

Fourier coefficients

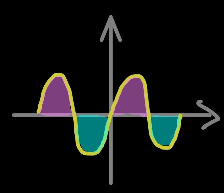
Example: $V = U = \mathrm{Span}\left(x \mapsto \frac{1}{\sqrt{2}} , \; x \mapsto \cos(x) , \; x \mapsto \cos(2x) , \; x \mapsto \sin(x) \right)$

(subspace in $\mathcal{F}(\mathbb{R})$)

with inner product: $\langle f , g \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) g(x) \, dx$

We get: $\langle x \mapsto \cos(x) , x \mapsto \cos(x) \rangle = \frac{1}{\pi} \cdot \underbrace{\int_{-\pi}^{\pi} (\cos(x))^2 \, dx}_{\pi} = 1$

$$\langle x \mapsto \cos(x) , x \mapsto \sin(x) \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(x) \sin(x)}_{\text{odd function}} \, dx$$



$$= 0$$

$\vdots$

$$\Longrightarrow \quad \mathcal{B} = \left(x \mapsto \frac{1}{\sqrt{2}} , \; x \mapsto \cos(x) , \; x \mapsto \cos(2x) , \; x \mapsto \sin(x) \right) \quad \text{ONB}$$

Take u with $u(x) = (\sin(x))^2$ (actually $u \in V$)

Calculate:

$$\langle b_1 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} (\sin(x))^2 \, dx = \frac{1}{\sqrt{2}}$$

$$\langle b_2 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \cos(x) (\sin(x))^2 \, dx = \frac{1}{\pi} \; \frac{1}{3} (\sin(x))^3 \Big|_{-\pi}^{\pi} = 0$$

$$\langle b_3 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \cos(2x) (\sin(x))^2 \, dx \underset{\text{longer calculation}}{=} -\frac{1}{2}$$

$$\langle b_4 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (\sin(x))^3 \, dx = 0$$

$$\Longrightarrow \quad u = b_1 \langle b_1 , u \rangle + b_3 \langle b_3 , u \rangle$$

$$(\sin(x))^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \cos(2x) \cdot \left(-\frac{1}{2} \right) = \frac{1}{2} \cdot \left(1 - \cos(2x) \right)$$