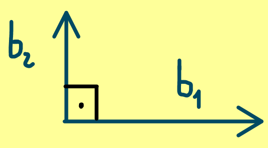
The Bright Side of Mathematics

Abstract Linear Algebra – Part 18

Assumption: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

Idea: Choose a nice basis $(b_1, b_2, \dots, b_k)$ of U:



$$\langle b_1, b_2 \rangle = 0$$

$$\langle b_1, b_1 \rangle = \|b_1\|^2 = 1 \; , \; \langle b_2, b_2 \rangle = 1$$

Notation: $\langle b_i, b_j \rangle = \underset{\text{Kronecker delta}}{\delta_{ij}} := \begin{cases} 1 \; , & i = j \\ 0 \; , & i \neq j \end{cases}$

Orthogonal projection: For $x \in V$: $x = \underset{\in U}{x|_U} + \underset{\in U^\perp}{x|_{U^\perp}}$ can be calculated:

$\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U

$$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix} \rightsquigarrow \text{solving LES gives } x|_U$$

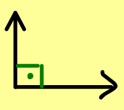
Gramian matrix

$$G(\mathcal{B}) = \begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix} \overset{\substack{\text{nice} \\ \text{basis}}}{=} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\text{identity matrix}}$$

$$\Rightarrow \quad x|_U = \sum_{j=1}^{k} b_j \langle b_j, x \rangle$$

Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

A family $(b_1, b_2, \dots, b_m)$ (with $b_j \in U$) is called:



- orthogonal system (OS) if $\langle b_i, b_j \rangle = 0$ for all $i \neq j$
- orthonormal system (ONS) if $\langle b_i, b_j \rangle = \delta_{ij}$
- orthogonal basis (OB) if it's an OS and a basis of U
- orthonormal basis (ONB) if it's an ONS and a basis of U

Example: $\mathbb{R}^3$ with standard inner product, $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ ONB of $\mathbb{R}^3$.