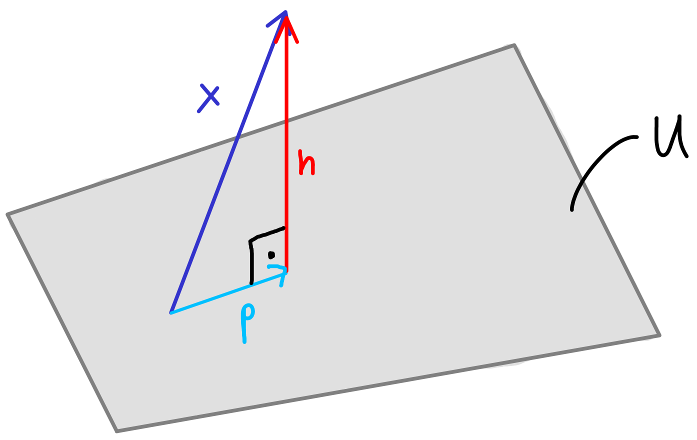
Abstract Linear Algebra – Part 17

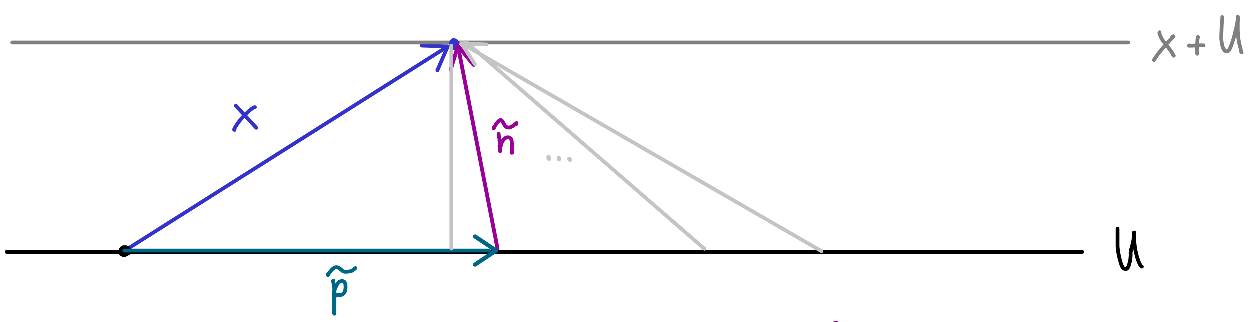
V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.



$$x = \underbrace{p}_{\in U} + \underbrace{n}_{\in U^\perp}$$

$$= x|_U + x|_{U^\perp}$$

Simplified picture: What is the distance between U and $x + U$?



$$x = \underbrace{\tilde{p}}_{\in U} + \tilde{n}$$

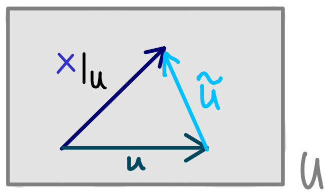
Approximation formula:

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

For $x \in V$: $\quad \operatorname{dist}(x, U) := \inf \left\{ \| x - u \| \;\middle|\; u \in U \right\} = \| x - \underbrace{x|_U}_{\text{orthogonal projection}} \|$

Recall: $\underbrace{\|x\|}_{\text{norm of } x} := \sqrt{\langle x, x \rangle}$

Proof: For all $u \in U$: $\|x - u\|^2 = \Big\| \underbrace{(x - x|_U)}_{n} + \underbrace{(x|_U - u)}_{=:\, \tilde{u} \in U} \Big\|^2$



normal component of x with respect to U

$$= \langle n + \tilde{u} \,,\, n + \tilde{u} \rangle$$

$$= \langle n, n \rangle + \underbrace{\langle n, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, n \rangle}_{=0} + \langle \tilde{u}, \tilde{u} \rangle$$

$n \in U^\perp$

$$= \|n\|^2 + \underbrace{\|\tilde{u}\|^2}_{\geq 0} \;\geq\; \|n\|^2$$

$$\Longrightarrow \inf\big\{ \|x - u\| \;\big|\; u \in U \big\} \geq \|n\|$$

We have equality $\iff$ $\hat{u} = 0$ $\iff$ $u = x|_U$ $\square$