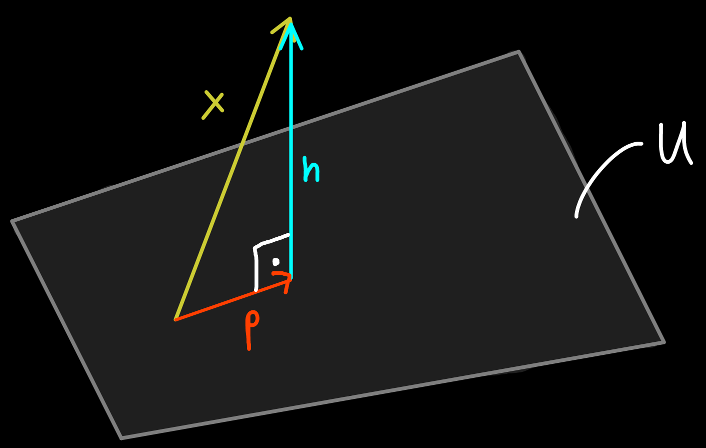
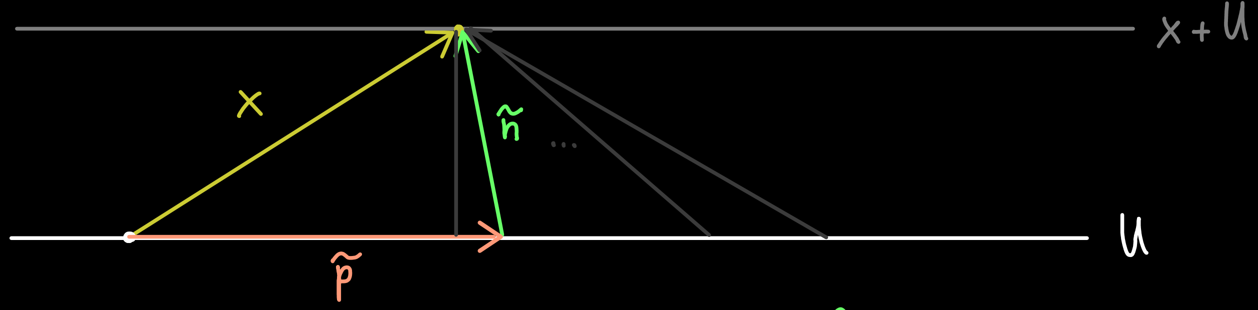
Abstract Linear Algebra – Part 17

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.



$$x = \underbrace{p}_{\in U} + \underbrace{n}_{\in U^\perp} = x|_U + x|_{U^\perp}$$

Simplified picture: What is the distance between U and $x + U$?



$$x = \underbrace{\tilde{p}}_{\in U} + \tilde{n}$$

Approximation formula:

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

For $x \in V$: $\quad \mathrm{dist}(x, U) := \inf \left\{ \|x - u\| \;\middle|\; u \in U \right\} = \|x - \underbrace{x|_U}_{}\|$

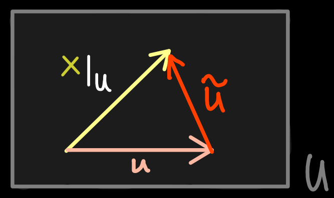
orthogonal projection

Recall: $\|x\| := \sqrt{\langle x, x \rangle}$

norm of x

Proof: For all $u \in U$: $\|x - u\|^2 = \|\underbrace{(x - x|_U)}_{n} + \underbrace{(x|_U - u)}_{=:\, \tilde{u} \in U}\|^2$

normal component of x with respect to U



$$= \langle n + \tilde{u} \,,\, n + \tilde{u} \rangle$$

$$= \langle n, n \rangle + \underbrace{\langle n, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, n \rangle}_{=0} + \langle \tilde{u}, \tilde{u} \rangle \qquad n \in U^{\perp}$$

$$= \|n\|^2 + \underbrace{\|\tilde{u}\|^2}_{\geq 0} \geq \|n\|^2$$

$$\Longrightarrow \inf\{\|x - u\| \mid u \in U\} \geq \|n\|$$

We have equality $\Longleftrightarrow$ $\hat{u} = 0$ $\Longleftrightarrow$ $u = x|_U$ $\square$