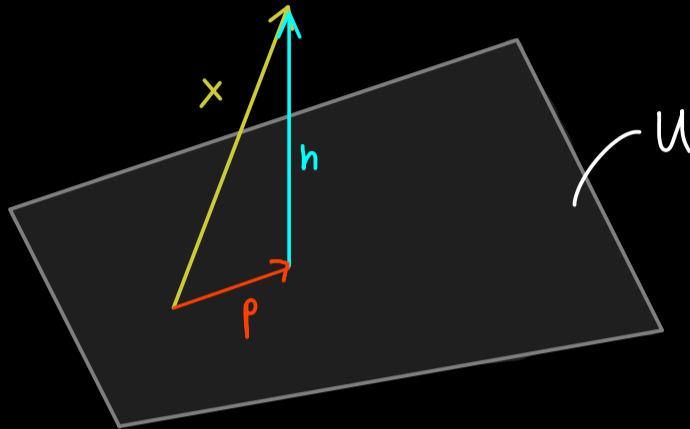


## Abstract Linear Algebra - Part 16

Orthogonal projection:

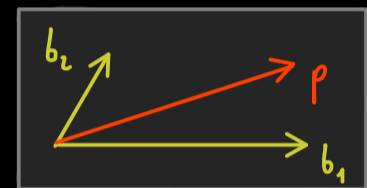
$V$   $\mathbb{F}$ -vector space, inner product  $\langle \cdot, \cdot \rangle$ ,  
 $U \subseteq V$   $k$ -dimensional subspace.



$$x = p + n \in U^\perp$$

Assume we have a basis  $\mathcal{B} = (b_1, b_2, \dots, b_k)$  of  $U$ .

$$p = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k \quad \text{for some } \lambda_1, \dots, \lambda_k \in \mathbb{F}$$



$$\begin{aligned} \text{For each basis vector } b_j : \langle b_j, x \rangle &= \langle b_j, p \rangle + \underbrace{\langle b_j, n \rangle}_{=0} \\ &= \langle b_j, \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k \rangle \\ &= \sum_{i=1}^k \lambda_i \langle b_j, b_i \rangle \end{aligned}$$

Let's rewrite these  $k$  linear equations:

$$\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \dots & \langle b_k, b_k \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

Gramian matrix  $G(\mathcal{B})$

$\leadsto$  solution gives us the orthogonal projection

Do we have a unique solution?  $G(\mathcal{B})$  invertible  $\iff \text{Ker}(G(\mathcal{B})) = \{0\}$

Let's prove  $\text{Ker}(G(\mathcal{B})) = \{0\}$ : Choose  $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \text{Ker}(G(\mathcal{B}))$

$$G(\mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \text{for all } j: \beta_1 \langle b_j, b_1 \rangle + \beta_2 \langle b_j, b_2 \rangle + \dots + \beta_k \langle b_j, b_k \rangle = 0$$

linearity

$$\Rightarrow \text{for all } j: \langle b_j, \underbrace{\sum_{i=1}^k \beta_i b_i}_{\gamma \in U} \rangle = 0$$

Proposition  
part 15

$$\Rightarrow \gamma \in U^\perp$$

$$U \cap U^\perp = \{0\} \Rightarrow \gamma = 0 \Rightarrow \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example:  $\mathbb{R}^3$  with standard inner product,  $U = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

$$x = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow G(\mathcal{B}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \dots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \dots & \langle b_2, b_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \dots & \langle b_k, b_k \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \left( \begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 1 \end{array} \right) \xrightarrow{2 \cdot \text{II}} \left( \begin{array}{cc|c} 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right)$$

$$\Rightarrow p = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow \left( \begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right) \rightsquigarrow \begin{matrix} \lambda_2 = -1 \\ \lambda_1 = 2 \end{matrix}$$