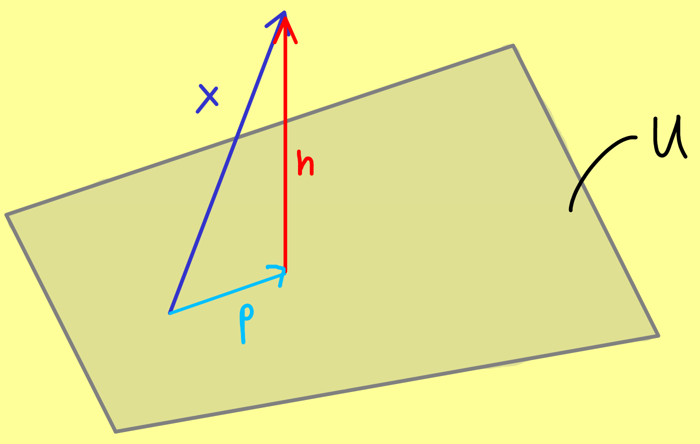
The Bright Side of Mathematics

Abstract Linear Algebra – Part 16

Orthogonal projection:

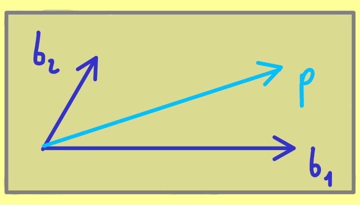


$x = \underbrace{p}_{\in U} + \underbrace{n}_{\in U^\perp}$

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,

$U \subseteq V$ k-dimensional subspace.

Assume we have a basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$ of U.



$p = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{F}$

For each basis vector b_j: $\langle b_j, x \rangle = \langle b_j, p \rangle + \underbrace{\langle b_j, n \rangle}_{=0}$

$$= \langle b_j, \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k \rangle$$

$$= \sum_{i=1}^{k} \lambda_i \langle b_j, b_i \rangle$$

Let's rewrite these k linear equations:

$$\underbrace{\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix}}_{\text{Gramian matrix } G(\mathcal{B})} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$\rightsquigarrow$ solution gives us the orthogonal projection

Do we have a unique solution? $G(\mathcal{B})$ invertible $\Longleftrightarrow$ $\mathrm{Ker}(G(\mathcal{B})) = \{0\}$

Let's prove $\mathrm{Ker}(G(\mathcal{B})) = \{0\}$: Choose $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \mathrm{Ker}(G(\mathcal{B}))$

$G(\mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies$ for all j: $\beta_1 \langle b_j, b_1 \rangle + \beta_2 \langle b_j, b_2 \rangle + \dots + \beta_k \langle b_j, b_k \rangle = 0$

linearity

$\implies$ for all j: $\langle b_j, \underbrace{\sum_{i=1}^{k} \beta_i b_i}_{y \in U} \rangle = 0$

$\overset{\text{Proposition part 15}}{\implies}$ $y \in U^\perp$

$\overset{U \cap U^\perp = \{0\}}{\implies}$ $y = 0$ $\implies$ $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Example: $\mathbb{R}^3$ with standard inner product, $U = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$x = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow G(\mathcal{B}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 1 \end{array} \right) \overset{2 \cdot \mathrm{II}}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right)$

$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right) \rightsquigarrow$ $\lambda_2 = -1$, $\lambda_1 = 2$

$\implies p = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$