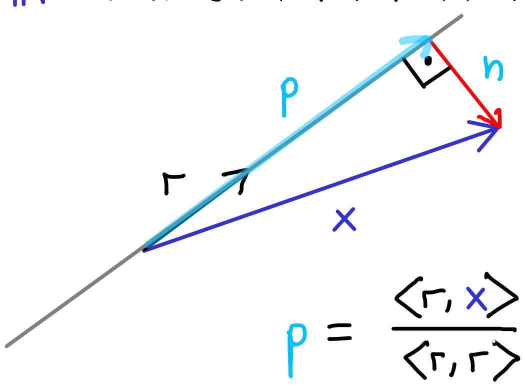
Abstract Linear Algebra - Part 15

Example: $\mathbb{R}^2$ with standard inner product.



$r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\hat{r} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (unit vector)

$x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

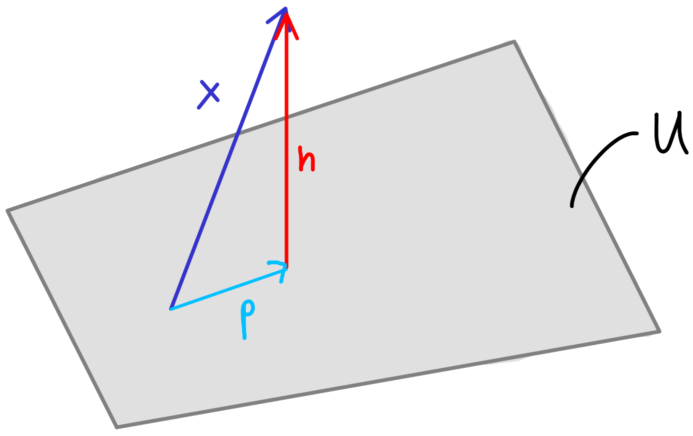
$$p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r = \langle \hat{r}, x \rangle \cdot \hat{r}$$

(orthogonal projection: $\hat{r} \langle \hat{r}, \cdot \rangle$)

$$\Rightarrow \quad p = \frac{1}{\sqrt{2}} (5+4) \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix}$$

$$\Rightarrow \quad n = x - p = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Generalization:



$$x = \underbrace{p}_{\in U} + \underbrace{n}_{\in U^\perp}$$

Important fact: $U \cap U^\perp = \{0\}$ for every subspace $U \subseteq V$

Proposition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

Let $U \subseteq V$ be a k-dimensional subspace, $\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U.

Then for $y \in V$: $\quad y \perp u$ for all $u \in U$

$$\Longleftrightarrow$$

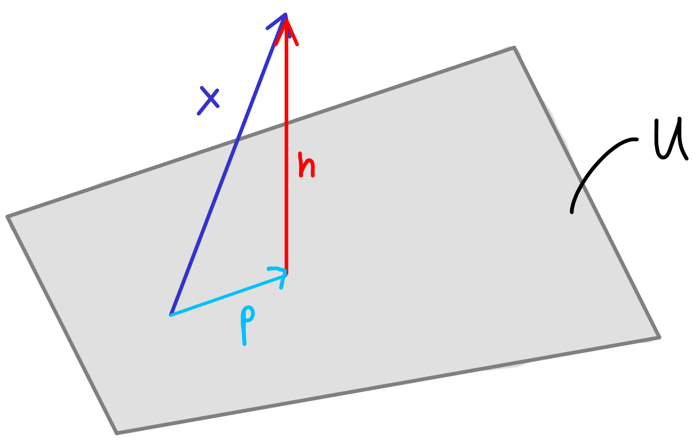
$$y \perp b_j \text{ for all } j \in \{1, 2, \dots, k\}$$

Proof: $(\Rightarrow)$ ✓ $\quad (\Leftarrow)$ We assume: $\langle y, b_j \rangle = 0$ for all $j \in \{1, 2, \dots, k\}$

$$\Rightarrow \sum_{j=1}^{k} \lambda_j \langle y, b_j \rangle = 0$$

$$\Rightarrow \Big\langle y, \sum_{j=1}^{k} \lambda_j b_j \Big\rangle = 0 \overset{\mathcal{B} \text{ basis}}{\Rightarrow} y \perp u \text{ for all } u \in U$$

Orthogonal projection onto a subspace:



V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,

$U \subseteq V$ k-dimensional subspace.

For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \in U^\perp$,

we call:

p orthogonal projection of x onto U

n normal component of x with respect to U