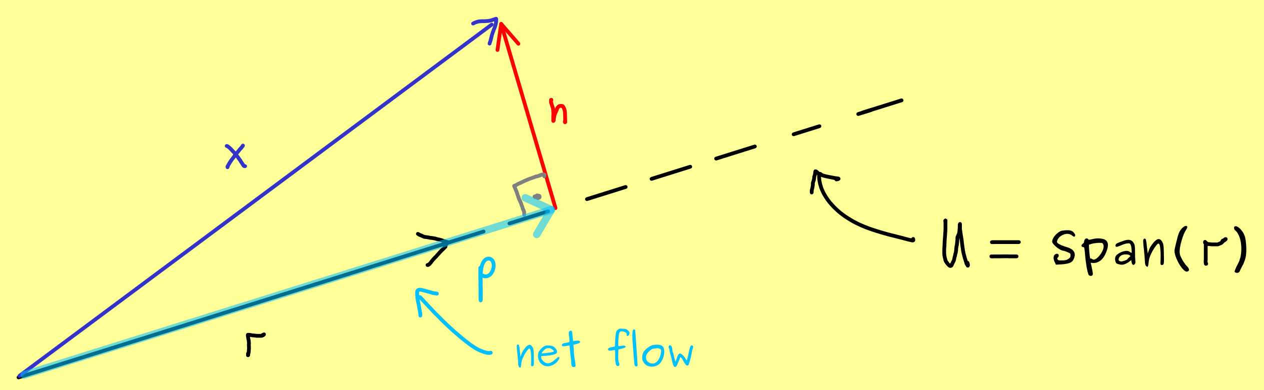
The Bright Side of Mathematics

Abstract Linear Algebra – Part 14



Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

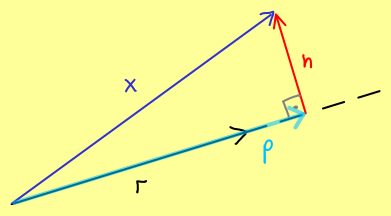
Let $U \subseteq V$ be a subspace with $U = \text{Span}(r)$, $r \neq 0$.

For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \perp r$,

we call:

p	orthogonal projection of x onto U
n	normal component of x with respect to U

Let's show the uniqueness: Assume $x = \underset{\in U}{p} + \underset{\in U^\perp}{n}$, $x = \underset{\in U}{\tilde{p}} + \tilde{n} \in U^\perp$



$$\Rightarrow \quad p + n = \tilde{p} + \tilde{n} \quad \Rightarrow \quad \underbrace{p - \tilde{p}}_{\in U} = \underbrace{\tilde{n} - n}_{\in U^\perp}$$

$$\Rightarrow \quad 0 = \langle p - \tilde{p}, \tilde{n} - n \rangle = \begin{cases} \langle p - \tilde{p}, p - \tilde{p} \rangle \\ \langle \tilde{n} - n, \tilde{n} - n \rangle \end{cases}$$

inner product is positive definite

$$\Rightarrow \quad p - \tilde{p} = 0 = \tilde{n} - n \quad \Rightarrow \quad p = \tilde{p} \quad \text{and} \quad n = \tilde{n}$$

Existence: $p \in U = \text{Span}(r) \Rightarrow p = \lambda \cdot r$ for $\lambda \in \mathbb{F}$

$$\langle r, x \rangle = \langle r, \lambda \cdot r + n \rangle = \lambda \langle r, r \rangle + \underbrace{\langle r, n \rangle}_{= 0}$$

$$\Rightarrow \quad \lambda = \frac{\langle r, x \rangle}{\langle r, r \rangle} \quad \rightsquigarrow \quad p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r \, , \quad n = x - p \quad \checkmark$$