Abstract Linear Algebra – Part 12

Recall: inner product on the $\mathbb{F}$-vector space V:

$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$ three properties!

For $V = \mathbb{F}^n$: $\langle y, x \rangle = \langle y, \underset{\substack{\uparrow \\ \text{positive definite matrix}}}{A} x \rangle_{\text{standard}}$

We use inner products for:

- measuring angles $\leftarrow$ Cauchy Schwarz inequality
- measuring lengths: $\underbrace{\|x\|}_{\text{norm of } x} := \sqrt{\langle x, x \rangle}$

Cauchy-Schwarz inequality: $\langle \cdot , \cdot \rangle$ inner product on the $\mathbb{F}$-vector space V.

Then: $|\langle y, x \rangle| \leq \|x\| \cdot \|y\|$ for all $x, y \in V$

and $|\langle y, x \rangle| = \|x\| \cdot \|y\| \iff x, y$ lin. dependent

Proof: (1) For $x = 0$: $\langle y, \underbrace{x}_{0 \cdot v} \rangle = 0 \cdot \langle y, v \rangle = 0$ and $\|x\| \cdot \|y\| = 0$

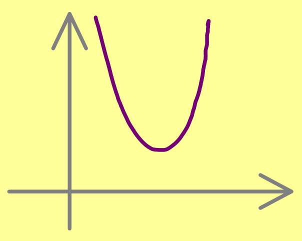
(2) For $x \neq 0$: Show: $\left|\langle y, \underbrace{\tfrac{x}{\|x\|}}_{\hat{x}} \rangle\right| \leq \|y\|$, $\|\hat{x}\| = 1$

For any $\lambda \in \mathbb{R}$: $0 \leq \langle y - \lambda \hat{x}, y - \lambda \hat{x} \rangle$

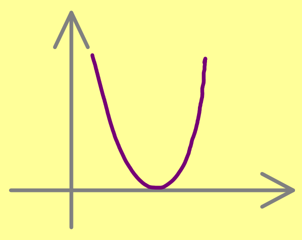
$$= \langle y, y \rangle - \lambda \underbrace{\langle \hat{x}, y \rangle}_{\bar{\alpha}} - \lambda \underbrace{\langle y, \hat{x} \rangle}_{\alpha} + \lambda^2 \langle \hat{x}, \hat{x} \rangle$$

$$= \lambda^2 + \lambda \cdot \underbrace{\left(-2 \cdot \mathrm{Re}(\langle y, \hat{x} \rangle)\right)}_{p} + \underbrace{\|y\|^2}_{q}$$

quadratic polynomial has zeros: $\lambda_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$



or



$$\Rightarrow \left(\frac{p}{2}\right)^2 - q \leq 0 \Rightarrow \mathrm{Re}(\langle y, \hat{x} \rangle)^2 \leq \|y\|^2$$

$$\Rightarrow |\mathrm{Re}(\langle y, \hat{x} \rangle)| \leq \|y\|$$ $\rightsquigarrow$ Cauchy-Schwarz $\mathbb{F} = \mathbb{R}$

For $\mathbb{F} = \mathbb{C}$: $\underbrace{e^{i\varphi}}_{c} \langle y, \hat{x} \rangle = |\langle y, \hat{x} \rangle|$

$$\| \quad |\mathrm{Re}(c \langle y, \hat{x} \rangle)| = |\mathrm{Re}(\langle y, \underbrace{c \hat{x}}_{\tilde{x}} \rangle)| \leq \|y\|$$