



Abstract Linear Algebra - Part 11

Example: In \mathbb{F}^2 :

$$\begin{aligned} \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle &= \bar{u}_1 v_1 + \bar{u}_1 v_2 + \bar{u}_2 v_1 + 4\bar{u}_2 v_2 \\ &= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}}_A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_{\text{standard}} \end{aligned}$$

→ check 3 rules of inner product

$$\hookrightarrow \langle x, x \rangle = \langle x, Ax \rangle_{\text{standard}} > 0 \quad \text{for } x \neq 0$$

Definition: $A \in \mathbb{F}^{n \times n}$ is called a positive definite matrix if:

- $A^* = A$ (selfadjoint/symmetric)
- $\langle x, Ax \rangle_{\text{standard}} > 0$ for all $x \in \mathbb{F}^n \setminus \{0\}$

Fact: If $A \in \mathbb{F}^{n \times n}$ is a positive definite matrix, then

$$\langle y, x \rangle := \langle y, Ax \rangle_{\text{standard}} \quad \text{defines an inner product in } \mathbb{F}^n.$$

Example: $\langle x, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} x \rangle_{\text{standard}} = \bar{x}_1 x_1 + \bar{x}_1 x_2 + \bar{x}_2 x_1 + 4\bar{x}_2 x_2$
 $= |x_1 + x_2|^2 + 3|x_2|^2 \geq 0$

$$\text{If } |x_1 + x_2|^2 + 3|x_2|^2 = 0 \quad \Rightarrow \quad |x_1 + x_2|^2 = 0 \quad \text{and} \quad \underbrace{|x_2|^2}_{\Rightarrow x_2 = 0} = 0$$

$\Rightarrow x_1 = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \text{ positive definite}$$

Proposition: For a selfadjoint matrix $A \in \mathbb{F}^{n \times n}$, the following claims are equivalent:

(a) A positive definite

(b) All eigenvalues of A are positive (> 0)

(c) After Gaussian elimination (without scaling and exchanging rows) only with row operations $Z_{i+\lambda j}$, (see part 37 of Linear Algebra) all pivots in the row echelon form are positive.

(d) The determinants of the so-called leading principal minors of A are positive.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$$H_1 = (a_{11}), \quad H_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$H_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \dots, \quad H_n = A$$

$$\det(H_1) > 0, \quad \det(H_2) > 0, \quad \dots, \quad \det(H_n) > 0$$

(Sylvester's criterion)

Example:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

(d) $\det(1) = 1 > 0$

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 4 - 1 = 3 > 0$$

(c) Gaussian elimination: $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{\text{II} - \text{I}} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$