The Bright Side of Mathematics

Abstract Linear Algebra – Part 11

Example: In $\mathbb{F}^2$:

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \overline{u_1}\cdot v_1 + \overline{u_1} v_2 + \overline{u_2} v_1 + 4\overline{u_2} v_2$$

$$= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_{\text{standard}}$$

$\rightsquigarrow$ check 3 rules of inner product

$\hookrightarrow \langle x, x \rangle = \langle x, Ax \rangle_{\text{standard}} > 0$ for $x \neq 0$

Definition: $A \in \mathbb{F}^{n\times n}$ is called a positive definite matrix if:

- $A^* = A$ (selfadjoint/symmetric)
- $\langle x, Ax \rangle_{\text{standard}} > 0$ for all $x \in \mathbb{F}^n \setminus \{0\}$

Fact: If $A \in \mathbb{F}^{n\times n}$ is a positive definite matrix, then

$$\langle y, x \rangle := \langle y, Ax \rangle_{\text{standard}}$$ defines an inner product in $\mathbb{F}^n$.

Example: $\left\langle x, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} x \right\rangle_{\text{standard}} = \overline{x_1}\cdot x_1 + \overline{x_1} x_2 + \overline{x_2} x_1 + 4\overline{x_2} x_2$

$$= |x_1 + x_2|^2 + 3\cdot|x_2|^2 \geq 0$$

If $|x_1 + x_2|^2 + 3\cdot|x_2|^2 = 0 \Rightarrow |x_1 + x_2|^2 = 0$ and $\underbrace{|x_2|^2 = 0}$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ positive definite

Proposition: For a selfadjoint matrix $A \in \mathbb{F}^{n\times n}$, the following claims are equivalent:

(a) A positive definite

(b) All eigenvalues of A are positive (>0)

(c) After Gaussian elimination (without scaling and exchanging rows) only with row operations $Z_{i+\lambda j}$, (see part 37 of Linear Algebra) all pivots in the row echelon form are positive.

(d) The determinants of the so-called leading principal minors of A are positive.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$H_1 = (a_{11})$, $H_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $H_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots, H_n = A$

$$\det(H_1) > 0, \ \det(H_2) > 0, \ \ldots, \ \det(H_n) > 0$$

(Sylvester's criterion)

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$

(d) $\det(1) = 1 > 0$

$\det\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 4 - 1 = 3 > 0$

(c) Gaussian elimination: $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{\text{II} - 1\text{I}} \begin{pmatrix} \boxed{1}^{>0} & 1 \\ 0 & \boxed{3}_{>0} \end{pmatrix}$