Abstract Linear Algebra - Part 10

Always: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

$$\overline{\alpha} := \begin{cases} \alpha , & \mathbb{F} = \mathbb{R} \\ \overline{\alpha} , & \mathbb{F} = \mathbb{C} \end{cases} \qquad \text{for } \alpha \in \mathbb{F}$$

$$A^* := \begin{cases} A^T , & \mathbb{F} = \mathbb{R} \\ A^* , & \mathbb{F} = \mathbb{C} \end{cases} \qquad \text{for } A \in \mathbb{F}^{m \times n}$$

Definition: $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$

is called an inner product on the $\mathbb{F}$-vector space V if:

(1) $\langle x, x \rangle \geq 0$ for all $x \in V$ (positive definite)

and $\langle x, x \rangle = 0 \implies x = 0$ (zero vector)

(2) $\langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle$ for all $x, \tilde{x}, y \in V$

$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in V$

(linear in the second argument)

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$ (conjugate symmetric)

Example: (a) For $u, v \in \mathbb{F}^n$, define:

$$\langle u, v \rangle_{\text{standard}} := \overline{u_1} \cdot v_1 + \overline{u_2} \cdot v_2 + \cdots + \overline{u_n} \cdot v_n = u^* v$$

(b) For $u, v \in \mathbb{F}^2$, define:

$$\langle u, v \rangle = \overline{u_1} \cdot v_2 + \overline{u_2} \cdot v_1 \rightsquigarrow \text{(2) and (3) satisfied}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -1 - 1 = -2 < 0 \rightsquigarrow \text{(1) not satisfied}$$

not an inner product!

(c) $\mathcal{P}([0,1], \mathbb{F})$ polynomial space, $p(x) = i\,x$ is in $\mathcal{P}([0,1], \mathbb{F})$

Define: $$\langle f, g \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx$$

Example: $$\langle p, p \rangle = \int_0^1 \overline{i x} \cdot i x \; dx = \int_0^1 x^2\, dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

$$\left(\sum_{i=1}^n \overline{u_i}\, v_i \rightsquigarrow \int_0^1 \overline{f} \cdot g \right)$$