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ON STEADY

The Bright Side of Mathematics

# Abstract Linear Algebra – Part 10

Always: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

$$\bar{\alpha} := \begin{cases} \alpha, & \mathbb{F} = \mathbb{R} \\ \bar{\alpha}, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } \alpha \in \mathbb{F}$$

$$A^* := \begin{cases} A^T, & \mathbb{F} = \mathbb{R} \\ A^*, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } A \in \mathbb{F}^{m \times n}$$

**Definition:** $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$

is called an inner product on the $\mathbb{F}$-vector space $V$ if:

(1) $\langle x, x \rangle \geq 0$ for all $x \in V$ (positive definite)

and $\langle x, x \rangle = 0 \Rightarrow x = 0$ (zero vector)

(2) $\langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle$ for all $x, \tilde{x}, y \in V$

$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in V$

(linear in the second argument)

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$ (conjugate symmetric)

**Example:** (a) For $u, v \in \mathbb{F}^n$, define:

$$\langle u, v \rangle_{\text{standard}} := \bar{u}_1 \cdot v_1 + \bar{u}_2 \cdot v_2 + \cdots + \bar{u}_n \cdot v_n = u^* v$$

(b) For $u, v \in \mathbb{F}^2$, define:

$$\langle u, v \rangle = \bar{u}_1 \cdot v_2 + \bar{u}_2 \cdot v_1 \rightsquigarrow \text{(2) and (3) satisfied}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -1 - 1 = -2 < 0 \rightsquigarrow \text{(1) not satisfied}$$

not an inner product!

(c) $\mathcal{P}([0,1], \mathbb{F})$ polynomial space, $p(x) = i x$ is in $\mathcal{P}([0,1], \mathbb{F})$

Define: $$\langle f, g \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx$$

Example: $$\langle p, p \rangle = \int_0^1 \overline{ix} \cdot i x \, dx = \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

$$\left( \sum_{i=1}^{n} \bar{u}_i v_i \rightsquigarrow \int_0^1 \bar{f} \cdot g \right)$$