Abstract Linear Algebra - Part 9

$$\underbrace{T_{\mathcal{B}\leftarrow\mathcal{C}}}_{\text{change-of-basis matrix}} \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}} = \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}}$$

$\Phi_{\mathcal{C}}(v)$ $\qquad$ $\Phi_{\mathcal{B}}(v)$

V vector space of dimension n

$\Downarrow$

$T_{\mathcal{B}\leftarrow\mathcal{C}}$ $\quad$ $(n \times n)$-matrix

invertible

Example:

$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$ $\qquad$ $\mathcal{C} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$

$\mathcal{E} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

standard basis

$V = \mathbb{R}^2$ $\quad$ $\mathcal{B}$ $\quad$ $\mathcal{E}$ $\quad$ $\mathcal{C}$

$$\begin{tikzcd}
 & V = \mathbb{R}^2 \arrow[dl, "\Phi_{\mathcal{B}}"', "\text{basis isomorphism}"] \arrow[d, "\Phi_{\mathcal{E}} = \mathrm{id}"] \arrow[dr, "\Phi_{\mathcal{C}}", "\text{basis isomorphism}"'] & \\
\mathbb{R}^2 \arrow[r, "T_{\mathcal{E}\leftarrow\mathcal{B}}"'] \arrow[rr, bend right, "T_{\mathcal{C}\leftarrow\mathcal{B}}"'] & \mathbb{R}^2 \arrow[r, shift right, "T_{\mathcal{C}\leftarrow\mathcal{E}}"'] & \mathbb{R}^2 \arrow[l, shift right, "T_{\mathcal{E}\leftarrow\mathcal{C}}"']
\end{tikzcd}$$

$\boxed{\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)}$

We already know:

$$T_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(b_1) & \Phi_{\mathcal{E}}(b_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$T_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(c_1) & \Phi_{\mathcal{E}}(c_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ c_1 & c_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

We can calculate:

$$T_{\mathcal{C}\leftarrow\mathcal{B}} = T_{\mathcal{C}\leftarrow\mathcal{E}}\, T_{\mathcal{E}\leftarrow\mathcal{B}}$$

$$= \underbrace{\left(T_{\mathcal{E}\leftarrow\mathcal{C}}\right)^{-1} T_{\mathcal{E}\leftarrow\mathcal{B}}}$$

calculate product immediately!

$$\underbrace{T_{\mathcal{E}\leftarrow\mathcal{C}}}_{= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}} X = \underbrace{T_{\mathcal{E}\leftarrow\mathcal{B}}}_{= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}}$$

$$\Longrightarrow \text{Solve} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right) \xrightarrow[\mathrm{II}\cdot\frac{1}{2}]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[\mathrm{I}-2\mathrm{II}]{} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$X = T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$