The Bright Side of Mathematics

Abstract Linear Algebra – Part 9

$$\underbrace{T_{\mathcal{B}\leftarrow\mathcal{C}}}_{\text{change-of-basis matrix}} \begin{pmatrix}\vdots\end{pmatrix}_{\mathcal{C}} = \begin{pmatrix}\vdots\end{pmatrix}_{\mathcal{B}}$$

$\Phi_{\mathcal{C}}(v)$ $\Phi_{\mathcal{B}}(v)$

V vector space of dimension n

$\Downarrow$

$T_{\mathcal{B}\leftarrow\mathcal{C}}$ $(n\times n)$-matrix

invertible

Example:

$\mathcal{B} = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right)$ $\mathcal{C} = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right)$

$\mathcal{E} = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$ standard basis

$$\begin{tikzcd}
& V=\mathbb{R}^2 \quad \mathcal{B}\ \ \mathcal{E}\ \ \mathcal{C} \arrow[dl, "\Phi_{\mathcal{B}}\ \text{basis isomorphism}"'] \arrow[d, "\Phi_{\mathcal{E}}=\mathrm{id}"] \arrow[dr, "\Phi_{\mathcal{C}}\ \text{basis isomorphism}"] & \\
\mathbb{R}^2 \arrow[r, "T_{\mathcal{E}\leftarrow\mathcal{B}}"'] \arrow[rr, bend right, "T_{\mathcal{C}\leftarrow\mathcal{B}}"'] & \mathbb{R}^2 \arrow[r, shift right, "T_{\mathcal{C}\leftarrow\mathcal{E}}"'] & \mathbb{R}^2 \arrow[l, shift right, "T_{\mathcal{E}\leftarrow\mathcal{C}}"']
\end{tikzcd}$$

$\boxed{\mathcal{B} = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right)}$

We already know:

$$T_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(b_1) & \Phi_{\mathcal{E}}(b_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$T_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(c_1) & \Phi_{\mathcal{E}}(c_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ c_1 & c_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

We can calculate:

$$T_{\mathcal{C}\leftarrow\mathcal{B}} = T_{\mathcal{C}\leftarrow\mathcal{E}}\, T_{\mathcal{E}\leftarrow\mathcal{B}}$$

$$= \underbrace{\left(T_{\mathcal{E}\leftarrow\mathcal{C}}\right)^{-1} T_{\mathcal{E}\leftarrow\mathcal{B}}}$$

calculate product immediately!

$$\underset{\begin{pmatrix}1&2\\0&2\end{pmatrix}}{T_{\mathcal{E}\leftarrow\mathcal{C}}}\; X = \underset{\begin{pmatrix}1&3\\2&4\end{pmatrix}}{T_{\mathcal{E}\leftarrow\mathcal{B}}}$$

$$\Longrightarrow \text{ Solve } \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right) \xrightarrow{\mathrm{II}\cdot\frac{1}{2}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{\mathrm{I}-2\mathrm{II}} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$X = T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$