Abstract Linear Algebra – Part 8

$$
\begin{tikzcd}
& V \arrow[dl, shift left, "\Phi_{\mathcal{B}}"'] \arrow[dr, shift left, "\Phi_{\mathcal{C}}"] & \\
\mathbb{F}^n \arrow[ur, shift left, "\Phi_{\mathcal{B}}^{-1}"'] \arrow[rr, shift right, "f"'] & & \mathbb{F}^n \arrow[ul, shift left, "\Phi_{\mathcal{C}}^{-1}"'] \arrow[ll, shift right, "f^{-1}"']
\end{tikzcd}
$$

vector space of dimension n

two bases $\mathcal{B}, \mathcal{C}$: $\mathcal{B} = (b_1, b_2, \dots, b_n)$, $\mathcal{C} = (c_1, c_2, \dots, c_n)$

change of basis: $f : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ linear!

old coordinates $\xrightarrow{f}$ new coordinates

What happens if we put $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ into f? $\rightsquigarrow$ $f(e_1) = \Phi_{\mathcal{C}}\big(\underbrace{\Phi_{\mathcal{B}}^{-1}(e_1)}_{b_1} \big) = \Phi_{\mathcal{C}}(b_1)$

We can see f as a matrix $f(x) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix}$

$$
T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(b_1) & \Phi_{\mathcal{C}}(b_2) & \cdots & \Phi_{\mathcal{C}}(b_n) \\ | & | & & | \end{pmatrix}
$$

transformation matrix
transition matrix
change-of-basis matrix
from $\mathcal{B}$ to $\mathcal{C}$

$$
\begin{tikzcd}
& v \in V \arrow[dl] \arrow[dr] & \\
\Phi_{\mathcal{B}}(v) & & \Phi_{\mathcal{C}}(v) \\
\begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}} \arrow[u, phantom, "\shortparallel"] \arrow[rr, "f"] & & \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}} \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}
$$

Remember!

$$
\boxed{\begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}} = T_{\mathcal{C} \leftarrow \mathcal{B}} \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}}}
$$

Fact: $\left(T_{\mathcal{C} \leftarrow \mathcal{B}} \right)^{-1} = T_{\mathcal{B} \leftarrow \mathcal{C}}$

<u>Example:</u> $V = \mathcal{P}_2(\mathbb{R})$ polynomials of degree ≤ 2

$m_0 : x \mapsto 1$

$m_1 : x \mapsto x$

$m_2 : x \mapsto x^2$

$$\mathcal{B} = (\overbrace{m_2}^{b_1}, \overbrace{m_1}^{b_2}, \overbrace{m_0}^{b_3})$$

$$\mathcal{C} = (\underbrace{m_2 - \tfrac{1}{2} m_1}_{c_1}, \underbrace{m_2 + \tfrac{1}{2} m_1}_{c_2}, \underbrace{m_0}_{c_3})$$

$T_{\mathcal{C} \leftarrow \mathcal{B}} \rightsquigarrow$ how to write b_j with a linear combination of $\mathcal{C}$

$T_{\mathcal{B} \leftarrow \mathcal{C}} \rightsquigarrow$ how to write c_j with a linear combination of $\mathcal{B}$

$\hookrightarrow$ column vectors $\Phi_{\mathcal{B}}(c_1) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$T_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{calculate inverse!}} T_{\mathcal{C} \leftarrow \mathcal{B}}$$