Abstract Linear Algebra - Part 4

We know: $\mathcal{P}_k(\mathbb{R}) := \{\text{polynomials with degree} \leq k\}$

$$\mathcal{P}_0(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{P}_1(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{P}_2(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \cdots \subseteq \mathcal{P}(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{F}(\mathbb{R})$$

Definition: V $\mathbb{F}$-vector space:

(a) For $v_1, \dots, v_k \in V$, $\alpha_1, \dots, \alpha_k \in \mathbb{F}$,

$$\sum_{j=1}^{k} \alpha_j v_j$$ is called a linear combination.

(b) For subset $M \subseteq V$:

$$\mathrm{Span}(M) := \{\text{all possible linear combinations with vectors from } M\}$$

$$\mathrm{Span}(\emptyset) := \{0\}$$

subspace in V

(c) A set $M \subseteq V$ is called a generating set of a subspace $U \subseteq V$ if

$$\mathrm{Span}(M) = U$$

(d) A set $M \subseteq V$ is called a linearly independent if for all $k \in \mathbb{N}$ and $v_j \in V$:

$$0 = \sum_{j=1}^{k} \alpha_j v_j \quad \Longrightarrow \quad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

(e) A set $M \subseteq V$ (or an ordered family $M = (v_1, \dots, v_k)$)

is called a basis of a subspace $U \subseteq V$ if M is generating and lin. independent.

(f) The number of elements in a basis of U is called the dimension of U

cardinality of M

$$\dim(U) \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$$

could be distinguished more

Example:

(1) $\dim(\mathcal{P}_0(\mathbb{R})) = 1$

basis $M = (x \mapsto 1)$

space of constant functions/polynomials $\mathbb{R} \to \mathbb{R}$

(2) $\dim(\mathcal{P}_2(\mathbb{R})) = 3$

basis $M = (x \mapsto 1, x \mapsto x, x \mapsto x^2)$

polynomials of degree ≤ 2

(3) $\dim(\mathcal{F}(\mathbb{R})) = \infty$

(4) $\dim(\mathbb{C}^{2\times 3}) = 6$