The Bright Side of Mathematics

Abstract Linear Algebra – Part 20

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

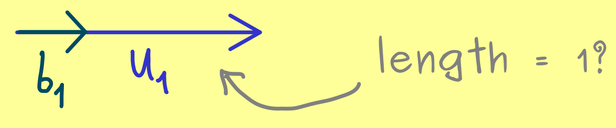
basis of U: $(u_1, u_2, \dots, u_k)$ $\rightsquigarrow$ ONB of U: $(b_1, b_2, \dots, b_k)$

Gram–Schmidt process/algorithm

$\langle b_i, b_j \rangle = \delta_{ij}$

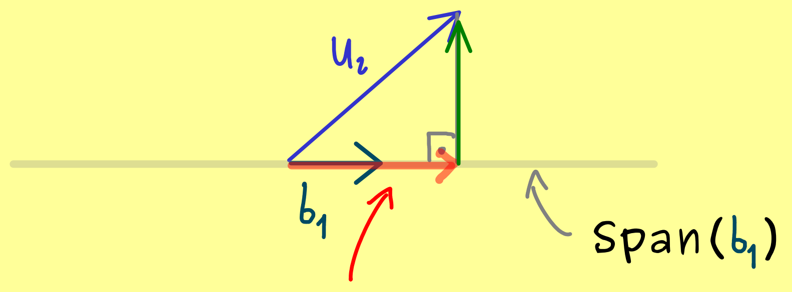
Gram–Schmidt orthonormalization:

(1) Normalize first vector:



$$b_1 := \frac{1}{\|u_1\|} \cdot u_1 \quad \text{where} \quad \|u_1\| := \sqrt{\langle u_1, u_1 \rangle}$$

(2) Next vector u_2:

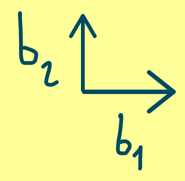


orthogonal projection of u_2 onto $\mathrm{Span}(b_1)$:

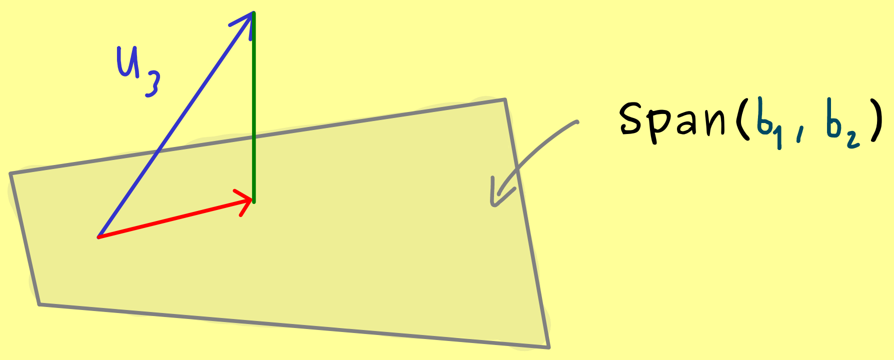
$$\hookrightarrow u_2\big|_{\mathrm{Span}(b_1)} = b_1 \langle b_1, u_2 \rangle$$

normal component: $\tilde{b}_2 = u_2 - b_1 \langle b_1, u_2 \rangle$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$



(3) Next vector u_3:



orthogonal projection of u_3 onto $\mathrm{Span}(b_1, b_2)$:

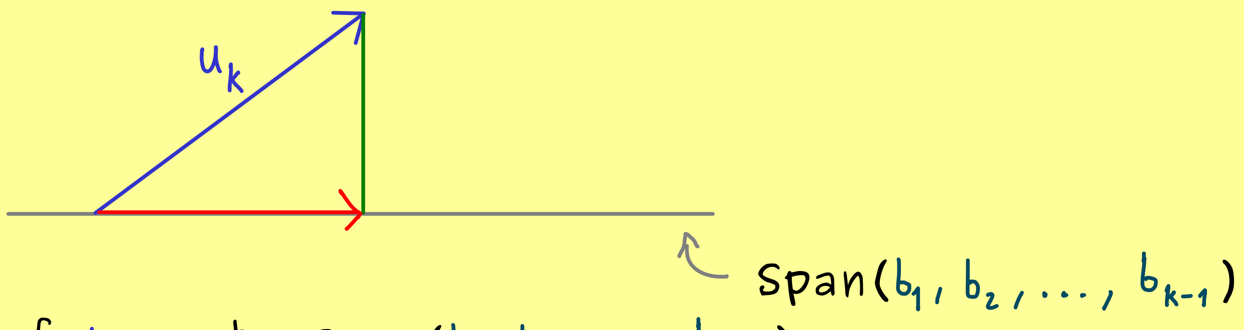
$$\hookrightarrow u_3\big|_{\mathrm{Span}(b_1,b_2)} := b_1 \langle b_1, u_3 \rangle + b_2 \langle b_2, u_3 \rangle$$

normal component: $\tilde{b}_3 = u_3 - b_1 \langle b_1, u_3 \rangle - b_2 \langle b_2, u_3 \rangle$

normalize it: $b_3 := \frac{1}{\|\tilde{b}_3\|} \tilde{b}_3$

⋮ continue!

(k) Next vector u_k:



orthogonal projection of u_k onto $\mathrm{Span}(b_1, b_2, \dots, b_{k-1})$

$$\hookrightarrow u_k\big|_{\mathrm{Span}(b_1, b_2, \dots, b_{k-1})} := \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$$

normal component: $\tilde{b}_k = u_k - \sum_{j=1}^{k-1} b_j \langle b_j, u_k \rangle$

normalize it: $b_k := \frac{1}{\|\tilde{b}_k\|} \tilde{b}_k \implies$ ONB of U: $(b_1, b_2, \dots, b_k)$