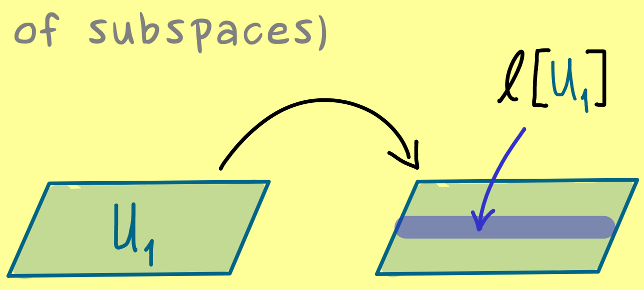
The Bright Side of Mathematics

Abstract Linear Algebra - Part 40

Proposition: Let V be an $\mathbb{F}$-vector space and $\ell : V \longrightarrow V$ be linear.
If $U_1, U_2 \subseteq V$ are subspaces such that

(1) $U_1 \oplus U_2 = V$ (direct sum of subspaces)

(2) U_1, U_2 are invariant under ℓ
$\left(\ell[U_1] \subseteq U_1,\ \ell[U_2] \subseteq U_2 \right)$



then ℓ has matrix representation in block diagonal form:

$$\ell_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} \boxed{} & 0 \\ 0 & \boxed{} \end{pmatrix}$$

square of size $\dim(U_1) \times \dim(U_1)$

square of size $\dim(U_2) \times \dim(U_2)$

Proof:

$$\begin{tikzcd} V \arrow[r, bend left, "\ell"] \arrow[d, "\Phi_{\mathcal{B}}"'] & V \arrow[d, "\Phi_{\mathcal{B}}"] \\ \mathbb{F}^n \arrow[r, "\ell_{\mathcal{B} \leftarrow \mathcal{B}}"] & \mathbb{F}^n \end{tikzcd}$$

$\alpha := \dim(U_1)$

Choose a suitable basis:
$$\mathcal{B} = (\underbrace{b_1, b_2, \dots, b_\alpha}_{\text{basis of } U_1}, \underbrace{b_{\alpha+1}, \dots, b_n}_{\text{basis of } U_2})$$

invariance of the subspaces implies: $\ell(b_j) \in \mathrm{Span}(b_1, b_2, \dots, b_\alpha)$ for all $j \in \{1, \dots, \alpha\}$

$\ell(b_k) \in \mathrm{Span}(b_{\alpha+1}, \dots, b_n)$ for all $k \in \{\alpha+1, \dots, n\}$

$$b_2 \overset{\Phi_{\mathcal{B}}}{\longmapsto} e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad \ell_{\mathcal{B} \leftarrow \mathcal{B}}\, e_j \in \mathrm{Span}(e_1, e_2, \dots, e_\alpha) \text{ for all } j \in \{1, \dots, \alpha\}$$
$$\ell_{\mathcal{B} \leftarrow \mathcal{B}}\, e_k \in \mathrm{Span}(e_{\alpha+1}, \dots, e_n) \text{ for all } k \in \{\alpha+1, \dots, n\}$$

$$\Longrightarrow \quad \ell_{\mathcal{B} \leftarrow \mathcal{B}} = \begin{pmatrix} \overbrace{\boxed{}}^{\alpha} \}\alpha & 0 \\ 0 & \underbrace{\boxed{}}_{n-\alpha} \}{n-\alpha} \end{pmatrix}$$

$\square$

Application: $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ eigenvalue of A. $N := A - \lambda \cdot \mathbb{1}$.

We know: (1) $\mathrm{Ker}(N^d)$, $\mathrm{Ran}(N^d)$ are invariant under A

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(2) $\mathbb{C}^n = \mathrm{Ker}(N^d) \oplus \mathrm{Ran}(N^d)$

Result:

$$A \approx \begin{pmatrix} \boxed{} \} & 0 \\ 0 & \boxed{} \end{pmatrix}$$

$\dim(\mathrm{Ker}(N^d))$