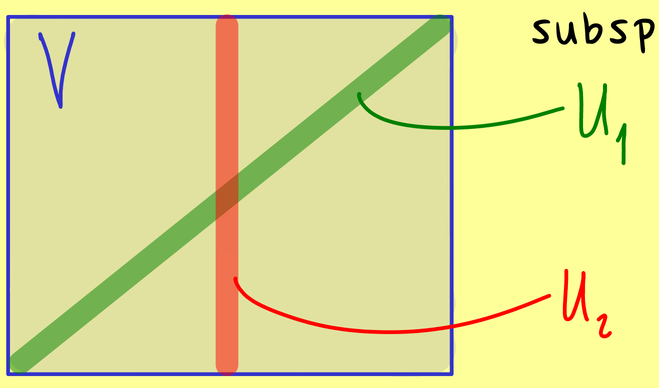
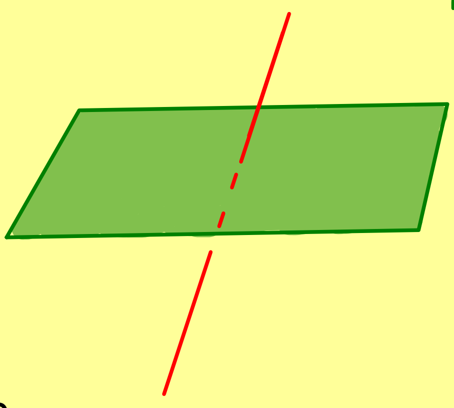
Abstract Linear Algebra - Part 39



$$U_1 + U_2 := \{ u_1 + u_2 \mid u_1 \in U_1 ,\ u_2 \in U_2 \}$$

Definition: Let V be an $\mathbb{F}$-vector space and U_1, U_2 be two subspaces of V.



If $U_1 \cap U_2 = \{0\}$, then $U_1 \oplus U_2 := \{ u_1 + u_2 \mid u_1 \in U_1 ,\ u_2 \in U_2 \}$

is called the direct sum of subspaces.

Example: $V = \mathbb{C}^2$. $\quad \mathrm{Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathrm{Span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbb{C}^2$

Properties:
- $U_1 \oplus U_2$ is a subspace of V
- $\dim(U_1 \oplus U_2) = \dim(U_1) + \dim(U_2)$

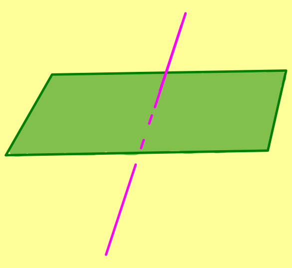
Our application of the direct sum: $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ eigenvalue of A.

$N := A - \lambda \mathbb{1}$. We get:

$$\{0\} \subsetneq \mathrm{Ker}(N^1) \subsetneq \mathrm{Ker}(N^2) \subsetneq \cdots \subsetneq \mathrm{Ker}(\underbrace{N^d}_{\text{Fitting index}}) = \mathrm{Ker}(N^{d+1}) = \cdots$$

$$\mathbb{C}^n \supsetneq \mathrm{Ran}(N^1) \supsetneq \mathrm{Ran}(N^2) \supsetneq \cdots \supsetneq \mathrm{Ran}(N^d) = \mathrm{Ran}(N^{d+1}) = \cdots$$

and $\mathbb{C}^n = \mathrm{Ker}(N^d) \oplus \mathrm{Ran}(N^d)$



Proof: Take $x \in \underbrace{\mathrm{Ker}(N^d)}_{N^d x = 0} \cap \underbrace{\mathrm{Ran}(N^d)}_{\text{there is } u \in \mathbb{C}^n \text{ with } N^d u = x \quad (*)}$

$$0 \overset{(*)}{=} N^d(N^d u) = N^{2d} u \implies u \in \mathrm{Ker}(N^{2d}) = \mathrm{Ker}(N^d)$$

$$0 = N^d u \overset{(*)}{=} x$$

$$\implies \mathrm{Ker}(N^d) \cap \mathrm{Ran}(N^d) = \{0\}$$

Moreover: $\dim\big(\mathrm{Ker}(N^d) \oplus \mathrm{Ran}(N^d)\big)$

$$= \dim\big(\mathrm{Ker}(N^d)\big) + \dim\big(\mathrm{Ran}(N^d)\big) = n$$

(n-dimensional subspace in $\mathbb{C}^n$) $\square$