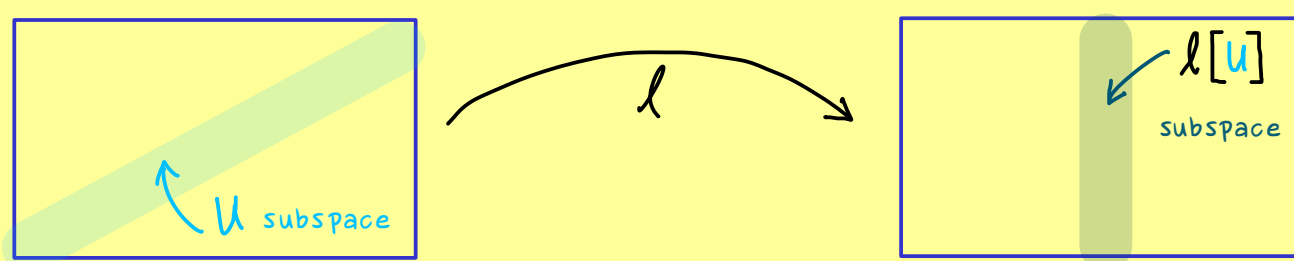




Abstract Linear Algebra - Part 38

Definition: Let V be an \mathbb{F} -vector space and $\ell: V \rightarrow V$ be a linear map.



A subspace U is called invariant under ℓ if $\ell[U] \subseteq U$.

Consequence: $\ell|_U: U \rightarrow U$ linear map

Important application: $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ eigenvalue of A .

We have generalized eigenspaces:

$$\{0\} \subsetneq \text{Ker}((A - \lambda \cdot \mathbb{1})^1) \subsetneq \text{Ker}((A - \lambda \cdot \mathbb{1})^2) \subsetneq \dots \subsetneq \text{Ker}((A - \lambda \cdot \mathbb{1})^d) \subsetneq \dots \subsetneq \mathbb{C}^n$$

Fitting index \downarrow
 $1 \leq d \leq n$

$$\mathbb{C}^n \supsetneq \text{Ran}((A - \lambda \cdot \mathbb{1})^1) \supsetneq \text{Ran}((A - \lambda \cdot \mathbb{1})^2) \supsetneq \dots \supsetneq \text{Ran}((A - \lambda \cdot \mathbb{1})^d) \supsetneq \dots \supsetneq \{0\}$$

Then: $\text{Ker}((A - \lambda \cdot \mathbb{1})^d)$ and $\text{Ran}((A - \lambda \cdot \mathbb{1})^d)$ are invariant under A

Proof: $(A - \lambda \cdot \mathbb{1})x = Ax - \lambda x$. So: $\begin{matrix} x \in U \\ (A - \lambda \cdot \mathbb{1})x \in U \end{matrix} \Rightarrow Ax = \underbrace{(A - \lambda \cdot \mathbb{1})x}_{\in U} + \underbrace{\lambda x}_{\in U}$

Define: $N := A - \lambda \cdot \mathbb{1}$

(1) Show $\text{Ker}(N^d)$ is invariant: $x \in \text{Ker}(N^d) \Rightarrow N^d(Nx) = N \underbrace{N^d x}_{=0} = 0 \Rightarrow Nx \in \text{Ker}(N^d)$

(2) Show $\text{Ran}(N^d)$ is invariant: $y \in \text{Ran}(N^d) \Rightarrow$ there is $x \in \mathbb{C}^n$ with $y = N^d x$
 $\Rightarrow Ny = N(N^d x) = N^d(\underbrace{Nx}_{= \tilde{x} \in \mathbb{C}^n}) \in \text{Ran}(N^d) \quad \square$