The Bright Side of Mathematics

Abstract Linear Algebra - Part 37

Let's fix: $A \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$ eigenvalue of A, $N := A - \lambda \cdot \mathbb{1}$

$E^{(j)} := \mathrm{Ker}(N^j)$ generalized eigenspace of rank j (w.r.t A and λ)

$R^{(j)} := \mathrm{Ran}(N^j)$ for $j = 0, 1, 2, 3, \dots$

We have a chain: $\{0\} = E^{(0)} \subsetneq E^{(1)} \subseteq E^{(2)} \subseteq \cdots \subseteq E^{(k)} = E^{(k+1)} = \cdots$

either dimension increases
or we have equality of sets

Fitting index: $\min \{ j \in \mathbb{N} \mid E^{(j)} = E^{(j+1)} \} =: d$

Chain picture: $\{0\} = E^{(0)} \subsetneq E^{(1)} \subsetneq \cdots \subsetneq E^{(j)} \subsetneq \cdots \subsetneq E^{(d)} = E^{(d+1)}$

N^j

$\mathbb{C}^n = R^{(0)} \supsetneq R^{(1)} \supsetneq \cdots \supsetneq R^{(j)} \supsetneq \cdots \supsetneq R^{(d)} \underset{(*)}{=} R^{(d+1)}$

Rank-nullity theorem: $\dim(R^{(j)}) + \dim(E^{(j)}) = n$ for all j

First result: $\min \{ j \in \mathbb{N} \mid R^{(j)} = R^{(j+1)} \} = d$, $N R^{(d)} = R^{(d+1)} \underset{(*)}{=} R^{(d)}$

$R^{(d+1)} = \{ N^{d+1}x \mid x \in \mathbb{C}^n \}$

linear map $R^{(d)} \xrightarrow{N} R^{(d)}$ is surjective $\Rightarrow$ bijective

$\Rightarrow R^{(d+2)} = N R^{(d+1)} = N R^{(d)} = R^{(d)} \overset{\text{inductively}}{\Rightarrow} R^{(d+j)} = R^{(d)}$ for all $j \in \mathbb{N}$

$\overset{\text{Rank-nullity theorem}}{\Longrightarrow} E^{(d+j)} = E^{(d)}$ for all $j \in \mathbb{N}$

The chain end with Fitting index!