The Bright Side of Mathematics

# Abstract Linear Algebra - Part 36

Jordan normal form:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_1 \end{matrix}} & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} \end{pmatrix}$$

Definition: Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$ be an eigenvalue of $A$.

Then: $\operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^k\right)$ is called the generalized eigenspace of rank $k$

An element $x \in \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^k\right) \setminus \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^{k-1}\right)$

is called a generalized eigenvector of rank $k$

Note: If $x$ generalized eigenvector of rank $k$, then

$$(A - \lambda \cdot \mathbb{1})^{k-1} x \in \operatorname{Ker}(A - \lambda \cdot \mathbb{1}) \setminus \{0\}.$$

We get a chain:

$$\{0\} = \operatorname{Ker}\Big(\underbrace{(A - \lambda \cdot \mathbb{1})^0}_{\mathbb{1}}\Big) \overset{\text{jump in dimension}}{\subsetneq} \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^1\right) \subseteq \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^2\right)$$

$$\cdots \subseteq \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^k\right) \subseteq \operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^{k+1}\right) \subseteq \cdots$$

$$\cdots \subseteq \mathbb{C}^n$$

Example: $A = \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} \rightsquigarrow \lambda = 2$ only eigenvalue $\rightsquigarrow A - \lambda \cdot \mathbb{1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^1\right) = \operatorname{Ker} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ← one-dimensional

$\operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^2\right) = \operatorname{Ker} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ← two-dimensional

$\operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^3\right) = \operatorname{Ker} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ← three-dimensional

$\operatorname{Ker}\left((A - \lambda \cdot \mathbb{1})^4\right)$ ← three-dimensional $\rightsquigarrow$ Fitting index is $3$