The Bright Side of Mathematics

Abstract Linear Algebra - Part 35

First thing to remember: Every square matrix $A \in \mathbb{C}^{n \times n}$ is similar to a so-called Jordan normal form $J \in \mathbb{C}^{n \times n}$.

triangular matrix with eigenvalues of A on the main diagonal

So: $A = X J X^{-1}$

If A is diagonalizable, then J is a diagonal matrix.

Example: (a) $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \rightsquigarrow 2, 3$ are the eigenvalues $\rightsquigarrow$ diagonalizable

(b) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \rightsquigarrow 2$ is the only eigenvalue

$\operatorname{Ker}(A - 2 \cdot \mathbb{1}) = \operatorname{Ker}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow$ all eigenvectors only span a one-dimensional space

$\rightsquigarrow$ not diagonalizable

Definition: A square matrix $J \in \mathbb{C}^{n \times n}$ is called a Jordan normal form if it can be written in block form:

$$J = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & J_3 & & \\ & & & \ddots & \\ & & & & J_r \end{pmatrix}, \quad 1 \leq r \leq n$$

The square matrices J_i are called Jordan blocks and they are in block form:

$$J_i = \begin{pmatrix} J_{i,1} & & & \\ & J_{i,2} & & \\ & & \ddots & \\ & & & J_{i,m} \end{pmatrix}$$

where the square matrices $J_{i,\ell}$ are called Jordan boxes.

They have a special form:

$$J_{i,\ell} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \lambda_i & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}$$

size can vary, also 1x1 possible

Example:

$$J = \begin{pmatrix} 4 & 1 & & & & & & & \\ & 4 & 1 & & & & & & \\ & & 4 & & & & & & \\ & & & 4 & 1 & & & & \\ & & & & 4 & & & & \\ & & & & & -3 & 1 & & \\ & & & & & & -3 & & \\ & & & & & & & -3 & \\ & & & & & & & & -3 \end{pmatrix}$$

$J_{1,1}$, $J_{1,2}$

J_1 (contains 2 Jordan boxes)

J_2 (contains 3 Jordan boxes)

$J_{2,1}$, $J_{2,2}$, $J_{2,3}$