The Bright Side of Mathematics

Abstract Linear Algebra - Part 26

Matrix representations:

$$
\begin{tikzcd}
V \text{ with basis: } \mathcal{B} = (b_1, \dots, b_n) \arrow[r, "\ell \text{ linear}"] \arrow[rr, bend left, "k \circ \ell"] \arrow[d, shift left, "\Phi_\mathcal{B}"] & W \text{ with basis: } \mathcal{C} = (c_1, \dots, c_m) \arrow[r, "k \text{ linear}"] \arrow[d, shift left, "\Phi_\mathcal{C}"] & U \text{ with basis: } \mathcal{D} = (d_1, \dots, d_k) \arrow[d, shift left, "\Phi_\mathcal{D}"] \\
\mathbb{F}^n \text{ (standard basis)} \arrow[u, shift left, "\Phi_\mathcal{B}^{-1}"] \arrow[r, "\ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}"'] & \mathbb{F}^m \text{ (standard basis)} \arrow[u, shift left, "\Phi_\mathcal{C}^{-1}"] \arrow[r, "k_{\mathcal{D} \leftarrow \mathcal{C}} \in \mathbb{F}^{k \times m}"'] & \mathbb{F}^k \text{ (standard basis)} \arrow[u, shift left, "\Phi_\mathcal{D}^{-1}"]
\end{tikzcd}
$$

We get: $(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}}$ (matrix product)

Example: $\ell : \mathbb{R}^3 \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto (v_1 + v_2 + v_3) \cdot m_0 + (v_1 + v_2) \cdot m_1 + v_1 \cdot m_2$

with $m_0 : x \mapsto 1$, $m_k : x \mapsto x^k$

$k : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$, $p \mapsto \begin{pmatrix} p'(1) \\ p(1) - p''(1) \end{pmatrix}$

Picture:

$$
\begin{tikzcd}
\mathbb{R}^3, \ \mathcal{B} \arrow[r, bend left, "\ell"] & \mathcal{P}_2, \ \mathcal{C} \arrow[r, bend left, "k"] & \mathbb{R}^2, \ \mathcal{D}
\end{tikzcd}
$$

standard basis (m_0, m_1, m_2) standard basis

$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = \, ?$

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} \Phi_\mathcal{C}(\ell(b_1)) & \Phi_\mathcal{C}(\ell(b_2)) & \Phi_\mathcal{C}(\ell(b_3)) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$k_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Corollary: $(\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = (\ell_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}$

$n = m$

$$
\begin{tikzcd}
V \text{ with basis: } \mathcal{B} = (b_1, \dots, b_n) \arrow[r, bend left, "\ell \text{ linear}"] & W \text{ with basis: } \mathcal{C} = (c_1, \dots, c_m) \arrow[l, bend left, "\ell^{-1}"]
\end{tikzcd}
$$