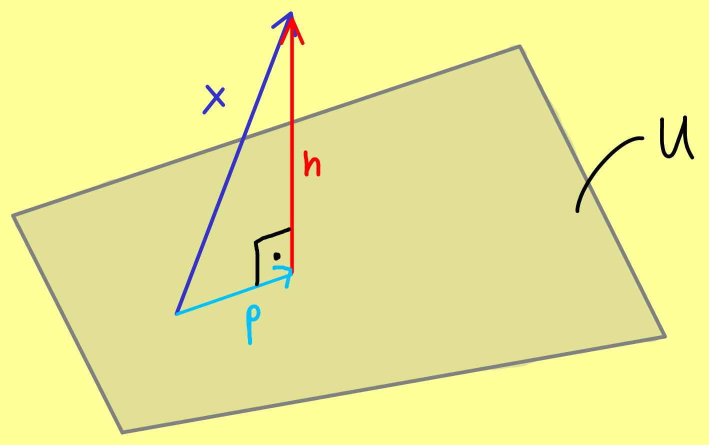
Abstract Linear Algebra – Part 17

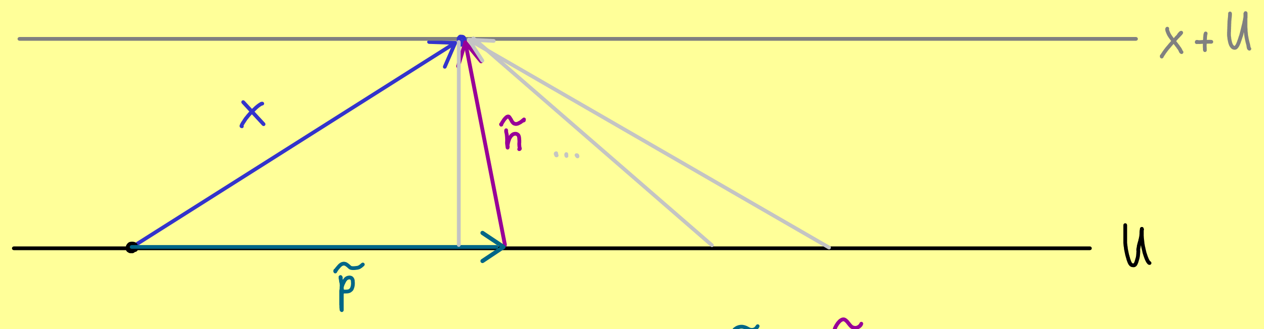
V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.



$$x = \underset{\in U}{p} + \underset{\in U^\perp}{n}$$

$$= x|_U + x|_{U^\perp}$$

Simplified picture: What is the distance between U and $x + U$?



$$x = \underset{\in U}{\tilde{p}} + \tilde{n}$$

Approximation formula:

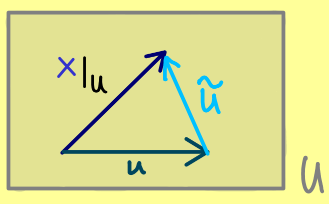
V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

For $x \in V$: $\operatorname{dist}(x, U) := \inf \{ \| x - u \| \mid u \in U \} = \| x - \underbrace{x|_U}_{} \|$

orthogonal projection

$\boxed{\text{Recall: } \|x\| := \sqrt{\langle x, x \rangle} \text{ norm of } x}$

Proof: For all $u \in U$: $\| x - u \|^2 = \| \underbrace{(x - x|_U)}_{n} + \underbrace{(x|_U - u)}_{=: \tilde{u} \in U} \|^2$



normal component of x with respect to U

$$= \langle n + \tilde{u} , n + \tilde{u} \rangle$$

$$= \langle n, n \rangle + \underbrace{\langle n, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, n \rangle}_{=0} + \langle \tilde{u}, \tilde{u} \rangle \qquad n \in U^\perp$$

$$= \|n\|^2 + \underbrace{\|\tilde{u}\|^2}_{\geq 0} \geq \|n\|^2$$

$$\Longrightarrow \inf \{ \| x - u \| \mid u \in U \} \geq \|n\|$$

We have equality $\Longleftrightarrow$ $\tilde{u} = 0$ $\Longleftrightarrow$ $u = x|_U$ $\square$