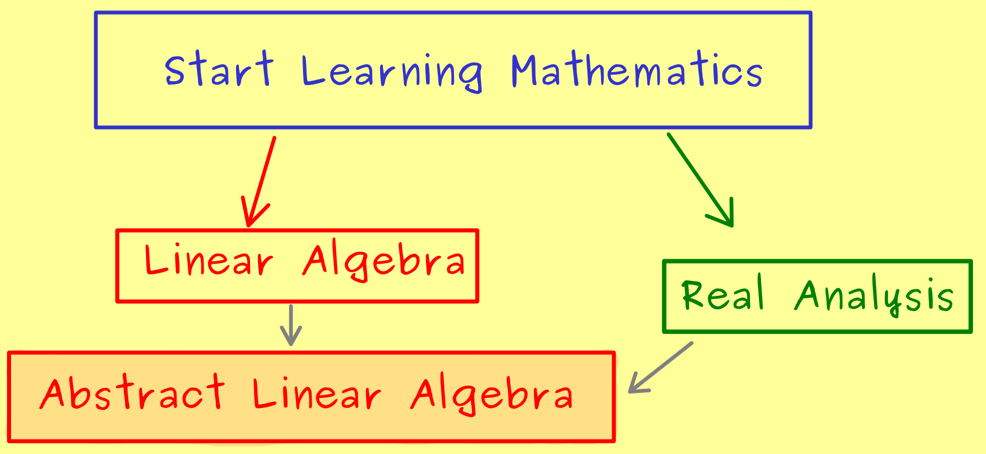
The Bright Side of Mathematics

The following pages cover the whole Abstract Linear Algebra course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Abstract Linear Algebra – Part 1

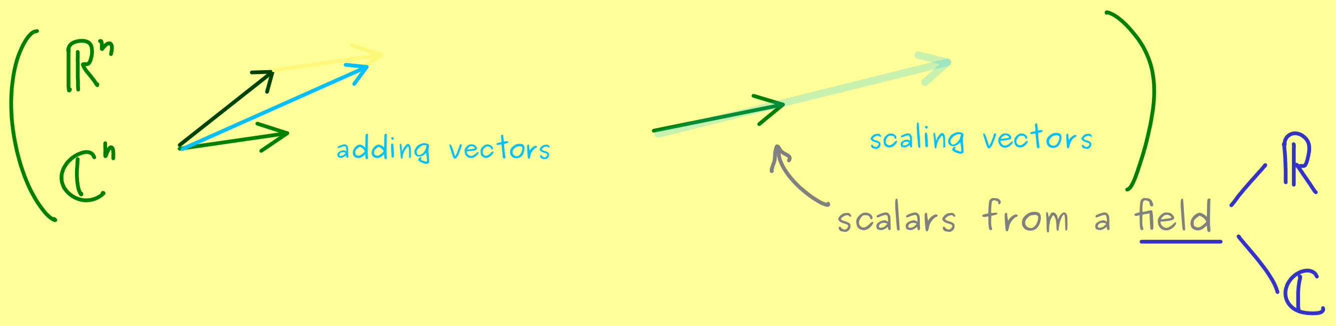
Prerequisites:



Content:

- general vector spaces
- general linear maps
- change of basis
- general inner products
- eigenvalue theory for linear maps

Definition: vector space $\boxed{V}$



Let $\mathbb{F}$ be a field (often $\mathbb{R}$ or $\mathbb{C}$).

A set $V \neq \emptyset$ together with two operations,

- vector addition $+ : V \times V \longrightarrow V$
- scalar multiplication $\cdot : \mathbb{F} \times V \longrightarrow V$

where the following eight rules are satisfied, is called an $\mathbb{F}$- vector space.

(a) $(V, +)$ is an abelian group:

(1) $u + (v + w) = (u + v) + w$ (associativity of $+$)

(2) $v + 0 = v$ with $0 \in V$ (neutral element)

(3) $v + (-v) = 0$ with $-v \in V$ (inverse elements)

(4) $v + w = w + v$ (commutativity of $+$)

(b) scalar multiplication is compatible:

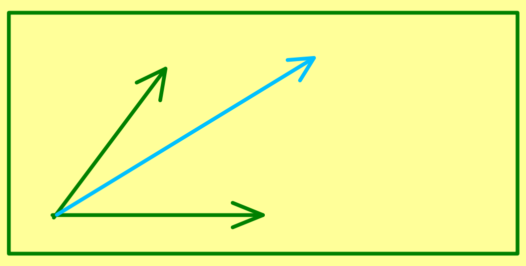
(5) $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v$

(6) $1 \cdot v = v$, $1 \in \mathbb{F}$ (multiplicative unit from the field)

(c) distributive laws:

(7) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$

(8) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ $\rightsquigarrow$ abstract vector space



Abstract Linear Algebra – Part 2

vector space: $\boxed{\begin{array}{l} \text{set} \\ \hookrightarrow \text{addition } + \\ \hookrightarrow \text{scalar multiplication } \cdot \end{array}}$ ← field $\mathbb{F}$ (important cases: $\mathbb{R}, \mathbb{C}$)

$\boxed{\text{8 rules for } + \text{ and } \cdot}$

Examples: (a) The space of matrices $\mathbb{C}^{m \times n}$ with matrix addition and scaling:

complex vector space (see: Linear Algebra – Part 11 and 58)

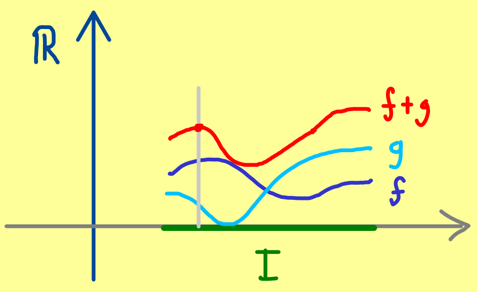
(b) Function space. Consider a set I and functions $f: I \longrightarrow \mathbb{R}$.

Then $\mathcal{F}(I) := \{ f: I \longrightarrow \mathbb{R} \}$ defines a real vector space:

- vector addition $f + g$ defined by:

$$(f+g)(x) := f(x) + g(x)$$

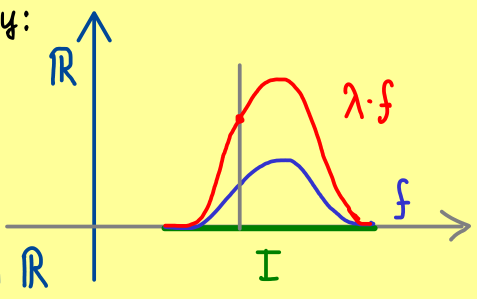
addition in $\mathbb{R}$



- scalar multiplication $\lambda \cdot f$ defined by:

$$(\lambda \cdot f)(x) := \lambda \cdot f(x)$$

multiplication in $\mathbb{R}$



↳ check 8 rules!

(c) Space of polynomials: $\mathcal{P}(\mathbb{R}) := \{ p: \mathbb{R} \to \mathbb{R} \text{ polynomial function} \}$

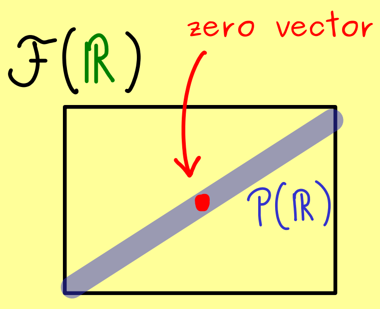
↳ $p(x) = a_n x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 x^1 + a_0$

$p_1 + p_2$, $\lambda \cdot p$ defined as before

$\Longrightarrow$ real vector space

We see: $\mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$

linear subspace in $\mathcal{F}(\mathbb{R})$



Abstract Linear Algebra – Part 3

$\boxed{\text{set + 8 rules}}$ V $\mathbb{F}$-vector space

↳ for example: space of functions

↳ zero vector $o \in V$

Question: $0 \cdot v = o$ ← zero vector , $(-1)\cdot v = -v$ for $v \in V$?

↑ zero in $\mathbb{F}$

Proof: $0 \cdot v = (0+0)\cdot v \overset{(8)}{=} 0\cdot v + 0 \cdot v$ ↙ associativity (1)

$$\overset{(3)}{\Rightarrow} 0\cdot v + (-(0\cdot v)) = 0\cdot v + \underbrace{\left(0\cdot v + (-(0\cdot v))\right)}_{=\,o}$$

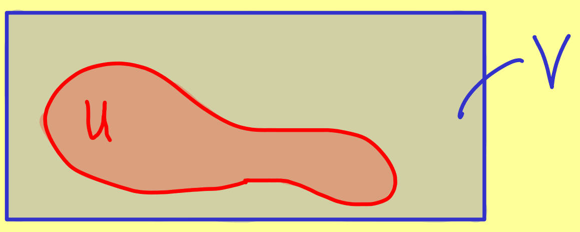
$$\overset{(3)}{\Rightarrow} o = 0 \cdot v \quad \checkmark$$

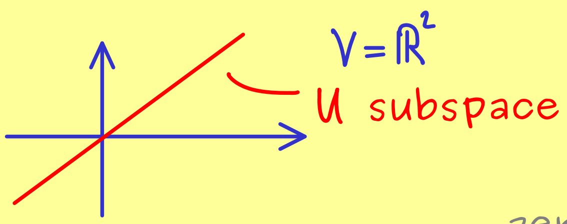
$$= (1+(-1))\cdot v \overset{(8)}{=} \underbrace{1\cdot v}_{(6)\nearrow\, v} + (-1)\cdot v$$

$$\overset{(3)}{\Rightarrow} -v + o = \underbrace{-v + v}_{=\,o} + (-1)\cdot v \quad \Rightarrow \quad -v = (-1)\cdot v \quad \checkmark$$

Linear subspace:

- vector space inside another one



- 

- $\mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$
  - zero function lies in $\mathcal{P}(\mathbb{R})$
  - adding two polynomials gives polynomial
  - scaling polynomial gives polynomial

Definition: V $\mathbb{F}$-vector space, $U \subseteq V$. If

(a) $o \in U$,

(b) $u, v \in U \Rightarrow u + v \in U$,

(c) $u \in U$, $\lambda \in \mathbb{F} \Rightarrow \lambda \cdot u \in U$,

then U is also an $\mathbb{F}$-vector space. We call it a linear subspace of V.

Example: $\mathcal{P}_2(\mathbb{R})$ polynomials with degree ≤ 2 $\quad (x \mapsto 4x^2 + x \ ,\ x \mapsto 8x + 1)$

$$\Rightarrow \mathcal{P}_2(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}) \quad \text{subspace}$$

Abstract Linear Algebra – Part 4

We know: $\mathcal{P}_k(\mathbb{R}) := \{$ polynomials with degree $\leq k \}$

$$\mathcal{P}_0(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{P}_1(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{P}_2(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \cdots \subseteq \mathcal{P}(\mathbb{R}) \underset{\text{subspace}}{\subseteq} \mathcal{F}(\mathbb{R})$$

Definition: V $\mathbb{F}$-vector space:

(a) For $v_1, \dots, v_k \in V$, $\alpha_1, \dots, \alpha_k \in \mathbb{F}$,

$$\sum_{j=1}^{k} \alpha_j v_j$$ is called a linear combination.

(b) For subset $M \subseteq V$:

$$\mathrm{Span}(M) := \{ \text{all possible linear combinations with vectors from } M \}$$

$$\mathrm{Span}(\emptyset) := \{ o \}$$

subspace in V

(c) A set $M \subseteq V$ is called a generating set of a subspace $U \subseteq V$ if

$$\mathrm{Span}(M) = U$$

(d) A set $M \subseteq V$ is called a linearly independent if for all $k \in \mathbb{N}$ and $v_j \in M$:

$$o = \sum_{j=1}^{k} \alpha_j v_j \quad \Rightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_k = 0$$

(e) A set $M \subseteq V$ (or an ordered family $M = (v_1, \dots, v_k)$)

is called a basis of a subspace $U \subseteq V$ if M is generating and lin. independent.

(f) The number of elements in a basis of U is called the dimension of U

cardinality of M

$$\dim(U) \in \{0, 1, 2, 3, \dots\} \cup \{\infty\}$$

could be distinguished more

Example:

(1) $\dim(\mathcal{P}_0(\mathbb{R})) = 1$ basis $M = (x \mapsto 1)$

space of constant functions/polynomials $\mathbb{R} \to \mathbb{R}$

(2) $\dim(\mathcal{P}_2(\mathbb{R})) = 3$ basis $M = (x \mapsto 1, x \mapsto x, x \mapsto x^2)$

polynomials of degree ≤ 2

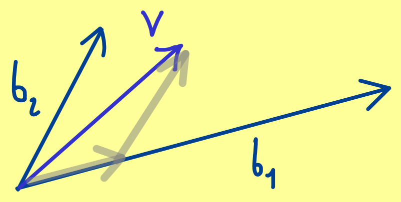
(3) $\dim(\mathcal{F}(\mathbb{R})) = \infty$

(4) $\dim(\mathbb{C}^{2\times 3}) = 6$ ($\mathbb{C}^{2\times 3}$ seen as a complex vector space)

Abstract Linear Algebra – Part 5

Coordinates with respect to a basis:

Assumptions: $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, V $\mathbb{F}$-vector space with $\dim(V) = n < \infty$, $\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V.



Then: each vector $v \in V$ can be uniquely written as: $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$ with $\alpha_j \in \mathbb{F}$

Definition: α_j are called the coordinates of v with respect to $\mathcal{B}$.

Remember: $v = \sum_{j=1}^{n} \alpha_j b_j \quad \xleftrightarrow{1:1} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{F}^n$

coordinate vector

Picture:

$$\begin{tikzcd}
\boxed{\text{abstract vector space } V \quad \text{fix basis } \mathcal{B}} \arrow[d, bend right, "\Phi_{\mathcal{B}}"'] \\
\boxed{\text{concrete vector space } \mathbb{F}^n \quad \text{standard basis}} \arrow[u, bend right, "\Phi_{\mathcal{B}}^{-1}"']
\end{tikzcd}$$

Define: $\Phi_{\mathcal{B}}(\alpha_1 b_1 + \cdots + \alpha_n b_n) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

$\Phi_{\mathcal{B}} : V \longrightarrow \mathbb{F}^n$ is a linear map:

$$\Phi_{\mathcal{B}}(v + w) = \Phi_{\mathcal{B}}(v) + \Phi_{\mathcal{B}}(w)$$

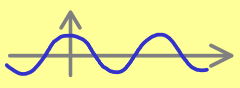
$$\Phi_{\mathcal{B}}(\lambda \cdot v) = \lambda \cdot \Phi_{\mathcal{B}}(v)$$

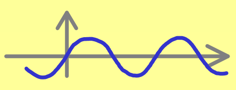
$\Phi_{\mathcal{B}}$ is called basis isomorphism

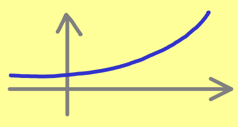
$\Phi_{\mathcal{B}}(b_j) = e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ canonical unit vector

Abstract Linear Algebra – Part 6

subset of $\mathcal{F}(\mathbb{R})$ given by:

cos: $\mathbb{R} \to \mathbb{R}$ $\rightsquigarrow$ 

sin: $\mathbb{R} \to \mathbb{R}$ $\rightsquigarrow$ 

exp: $\mathbb{R} \to \mathbb{R}$ $\rightsquigarrow$ 

$$U := \text{Span}(\cos, \sin, \exp)$$

Question: Is $(\cos, \sin, \exp)$ a basis of U? — generating ✓, linearly independent ?

We have to check: $\underbrace{\alpha_1 \cdot \cos + \alpha_2 \cdot \sin + \alpha_3 \cdot \exp = 0}_{\text{means:}} \Rightarrow \alpha_j = 0$ for all j

zero vector in $\mathcal{F}(\mathbb{R})$ ↳ $0 : \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$

$$\alpha_1 \cdot \cos(x) + \alpha_2 \cdot \sin(x) + \alpha_3 \cdot \exp(x) = 0(x)$$

for all $x \in \mathbb{R}$

$$\Rightarrow \begin{cases} \alpha_1 \cdot \cos(0) + \alpha_2 \cdot \sin(0) + \alpha_3 \cdot \exp(0) = 0 \\ \alpha_1 \cdot \cos(\frac{\pi}{2}) + \alpha_2 \cdot \sin(\frac{\pi}{2}) + \alpha_3 \cdot \exp(\frac{\pi}{2}) = 0 \\ \alpha_1 \cdot \cos(-2\pi \cdot 500) + \alpha_2 \cdot \sin(-2\pi \cdot 500) + \alpha_3 \cdot \exp(-2\pi \cdot 500) = 0 \end{cases}$$

$$\Rightarrow \left\{ \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix}}_{} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right.$$ system of linear equations

Since $\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e^{\pi/2} \\ 1 & 0 & e^{-1000\pi} \end{pmatrix} = e^{-1000\pi} + 0 + 0 - 1 - 0 - 0 < 0$,

the system of linear equations is uniquely solvable.

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0 \quad \Rightarrow \underbrace{(\cos, \sin, \exp)}_{= \mathcal{B}} \text{ basis of } U$$

Basis isomorphism: $\Phi_{\mathcal{B}} : U \to \mathbb{R}^3$,

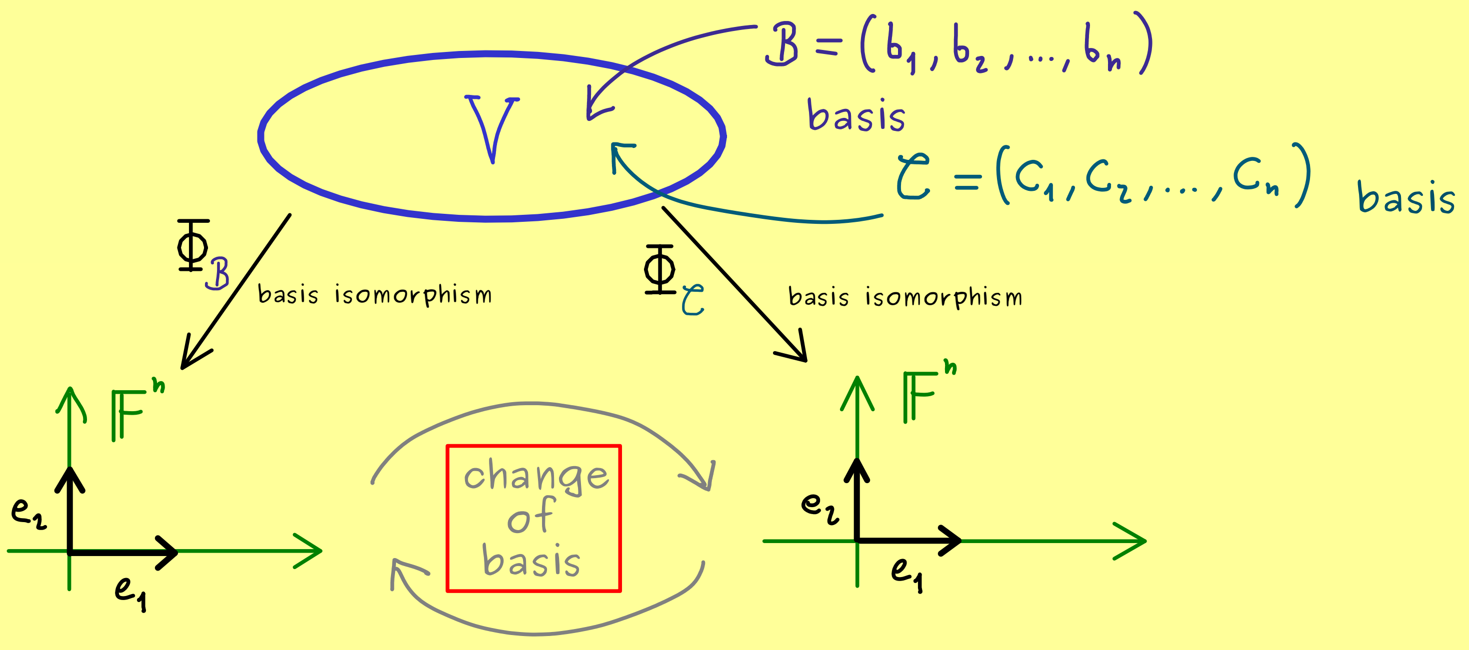
defined by $\Phi_{\mathcal{B}}(\cos) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\sin) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(\exp) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

What about $v : \mathbb{R} \to \mathbb{R}$, $v(x) = 7\cos(x) + 2\exp(x)$

$$\Phi_{\mathcal{B}}(v) = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$$

$\boxed{U \text{ is completely represented by } \mathbb{R}^3}$

Abstract Linear Algebra – Part 7



Recall: $\Phi_\mathcal{B} : V \longrightarrow \mathbb{F}^n$ given by $\Phi_\mathcal{B}(b_j) = e_j$ for all j

$\Phi_\mathcal{B}^{-1} : \mathbb{F}^n \longrightarrow V$ given by $\Phi_\mathcal{B}^{-1}(e_j) = b_j$ for all j

For each $v \in V$: $\quad v = \Phi_\mathcal{B}^{-1}\left(\begin{pmatrix} \text{coordinate} \\ \text{vector} \end{pmatrix}\right)$

Example: $\mathcal{P}_2(\mathbb{R})$ with basis $\mathcal{B} = (m_0, m_1, m_2)$ where $m_0(x) = 1$, $m_1(x) = x$, $m_2(x) = x^2$

For $p \in \mathcal{P}_2(\mathbb{R})$ given $p(x) = 3x^2 + 8x - 2$

$$p = (-2)\cdot m_0 + 8\cdot m_1 + 3\cdot m_2 = \Phi_\mathcal{B}^{-1}\left(\begin{pmatrix} -2 \\ 8 \\ 3 \end{pmatrix}\right)$$

coordinate vector

Another basis: $\mathcal{C} = (c_1, c_2, c_3)$ with $c_1 = m_0$, $c_2 = m_1$, c_3 polynomial

$c_3(x) = 3x^2 + 8x$

$$p = \Phi_\mathcal{C}^{-1}\left(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right)$$

coordinate vector

Old vs. new coordinates: $\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis, $\mathcal{C} = (c_1, c_2, \dots, c_n)$ basis

$$\Phi_\mathcal{B}(v) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \quad \longleftrightarrow \quad \Phi_\mathcal{C}(v) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$v = \beta_1 \cdot b_1 + \dots + \beta_n \cdot b_n \qquad v = \gamma_1 \cdot c_1 + \dots + \gamma_n \cdot c_n$$

$$\begin{tikzcd}
 & V \arrow[dl, shift right, "\Phi_\mathcal{B}"'] \arrow[dr, shift left, "\Phi_\mathcal{C}"] & \\
\mathbb{F}^n \arrow[ur, shift right, "\Phi_\mathcal{B}^{-1}"'] \arrow[rr, shift right, "f"'] & & \mathbb{F}^n \arrow[ul, shift left, "\Phi_\mathcal{C}^{-1}"] \arrow[ll, shift right, "f^{-1}"']
\end{tikzcd}$$

change of basis: $f : \mathbb{F}^n \longrightarrow \mathbb{F}^n$

linear!

We get: $f(x) = \Phi_\mathcal{C} \circ \Phi_\mathcal{B}^{-1}(x)$

Abstract Linear Algebra – Part 8

$$\begin{tikzcd} & V \ni \mathcal{B}, \mathcal{C} \arrow[dl, shift right, "\Phi_{\mathcal{B}}"'] \arrow[dr, shift left, "\Phi_{\mathcal{C}}"] & \\ \mathbb{F}^n \arrow[ur, shift right, "\Phi_{\mathcal{B}}^{-1}"'] \arrow[rr, shift right, "f"'] & & \mathbb{F}^n \arrow[ul, shift left, "\Phi_{\mathcal{C}}^{-1}"] \arrow[ll, shift right, "f^{-1}"'] \end{tikzcd}$$

vector space of dimension n

two bases $\mathcal{B} = (b_1, b_2, \dots, b_n)$, $\mathcal{C} = (c_1, c_2, \dots, c_n)$

change of basis: $f : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ linear!

old coordinates $\xrightarrow{f}$ new coordinates

What happens if we put $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ into f? $\rightsquigarrow$ $f(e_1) = \Phi_{\mathcal{C}}\big(\underbrace{\Phi_{\mathcal{B}}^{-1}(e_1)}_{b_1} \big) = \Phi_{\mathcal{C}}(b_1)$

We can see f as a matrix $f(x) = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix}$

$$T_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(b_1) & \Phi_{\mathcal{C}}(b_2) & \cdots & \Phi_{\mathcal{C}}(b_n) \\ | & | & & | \end{pmatrix}$$

transformation matrix
transition matrix
change-of-basis matrix
from $\mathcal{B}$ to $\mathcal{C}$

$$\begin{tikzcd} & v \in V \arrow[dl] \arrow[dr] & \\ \Phi_{\mathcal{B}}(v) = \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}} \arrow[rr, "f"] & & \Phi_{\mathcal{C}}(v) = \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}} \end{tikzcd}$$

Remember!

$$\boxed{\begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}} = T_{\mathcal{C} \leftarrow \mathcal{B}} \begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}}}$$

Fact: $\left(T_{\mathcal{C} \leftarrow \mathcal{B}} \right)^{-1} = T_{\mathcal{B} \leftarrow \mathcal{C}}$

Example: $V = \mathcal{P}_2(\mathbb{R})$ polynomials of degree ≤ 2

$m_0 : x \mapsto 1$
$m_1 : x \mapsto x$
$m_2 : x \mapsto x^2$

$\mathcal{B} = (\underbrace{m_2}_{b_1}, \underbrace{m_1}_{b_2}, \underbrace{m_0}_{b_3})$

$\mathcal{C} = (\underbrace{m_2 - \tfrac{1}{2} m_1}_{c_1}, \underbrace{m_2 + \tfrac{1}{2} m_1}_{c_2}, \underbrace{m_0}_{c_3})$

$T_{\mathcal{C} \leftarrow \mathcal{B}}$ $\rightsquigarrow$ how to write b_j with a linear combination of $\mathcal{C}$

$T_{\mathcal{B} \leftarrow \mathcal{C}}$ $\rightsquigarrow$ how to write c_j with a linear combination of $\mathcal{B}$

$\hookrightarrow$ column vectors $\Phi_{\mathcal{B}}(c_1) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_2) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $\Phi_{\mathcal{B}}(c_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$T_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\text{calculate inverse!}}{\rightsquigarrow} T_{\mathcal{C} \leftarrow \mathcal{B}}$$

Abstract Linear Algebra – Part 9

$$\underbrace{T_{\mathcal{B}\leftarrow\mathcal{C}}}_{\text{change-of-basis matrix}} \underbrace{\begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{C}}}_{\Phi_{\mathcal{C}}(v)} = \underbrace{\begin{pmatrix} \vdots \end{pmatrix}_{\mathcal{B}}}_{\Phi_{\mathcal{B}}(v)}$$

V vector space of dimension n

$\Downarrow$

$T_{\mathcal{B}\leftarrow\mathcal{C}}$ $(n \times n)$-matrix

invertible

Example:

$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)$ $\qquad$ $\mathcal{C} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$

$\mathcal{E} = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ standard basis

$$\begin{tikzcd}
 & V = \mathbb{R}^2 \quad \mathcal{B} \quad \mathcal{E} \quad \mathcal{C} \arrow[dl, "\Phi_{\mathcal{B}} \text{ basis isomorphism}"'] \arrow[d, "\Phi_{\mathcal{E}} = id"] \arrow[dr, "\Phi_{\mathcal{C}} \text{ basis isomorphism}"] & \\
\mathbb{R}^2 \arrow[r, "T_{\mathcal{E}\leftarrow\mathcal{B}}"'] \arrow[rr, bend right, "T_{\mathcal{C}\leftarrow\mathcal{B}}"'] & \mathbb{R}^2 \arrow[r, shift right, "T_{\mathcal{C}\leftarrow\mathcal{E}}"'] & \mathbb{R}^2 \arrow[l, shift right, "T_{\mathcal{E}\leftarrow\mathcal{C}}"']
\end{tikzcd}$$

$\boxed{\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right)}$

We already know:

$$T_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(b_1) & \Phi_{\mathcal{E}}(b_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$T_{\mathcal{E}\leftarrow\mathcal{C}} = \begin{pmatrix} | & | \\ \Phi_{\mathcal{E}}(c_1) & \Phi_{\mathcal{E}}(c_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ c_1 & c_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

We can calculate:

$$T_{\mathcal{C}\leftarrow\mathcal{B}} = T_{\mathcal{C}\leftarrow\mathcal{E}} \, T_{\mathcal{E}\leftarrow\mathcal{B}}$$

$$= \underbrace{\left(T_{\mathcal{E}\leftarrow\mathcal{C}} \right)^{-1} T_{\mathcal{E}\leftarrow\mathcal{B}}}_{\text{calculate product immediately!}}$$

$$\underbrace{T_{\mathcal{E}\leftarrow\mathcal{C}}}_{= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}} X = \underbrace{T_{\mathcal{E}\leftarrow\mathcal{B}}}_{= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}}$$

$$\Longrightarrow \text{Solve} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array} \right) \xrightarrow{\mathrm{II} \cdot \frac{1}{2}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array} \right)$$

$$\xrightarrow{\mathrm{I} - 2\mathrm{II}} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array} \right)$$

$$X = T_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$

Abstract Linear Algebra – Part 10

Always: $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$

$$\bar{\alpha} := \begin{cases} \alpha, & \mathbb{F} = \mathbb{R} \\ \bar{\alpha}, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } \alpha \in \mathbb{F}$$

$$A^* := \begin{cases} A^T, & \mathbb{F} = \mathbb{R} \\ A^*, & \mathbb{F} = \mathbb{C} \end{cases} \quad \text{for } A \in \mathbb{F}^{m \times n}$$

Definition: $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$

is called an inner product on the $\mathbb{F}$-vector space V if:

(1) $\langle x, x \rangle \geq 0$ for all $x \in V$ (positive definite)

and $\langle x, x \rangle = 0 \Rightarrow x = 0$ (zero vector)

(2) $\langle y, x + \tilde{x} \rangle = \langle y, x \rangle + \langle y, \tilde{x} \rangle$ for all $x, \tilde{x}, y \in V$

$\langle y, \lambda \cdot x \rangle = \lambda \cdot \langle y, x \rangle$ for all $\lambda \in \mathbb{F}, x, \tilde{x}, y \in V$

(linear in the second argument)

(3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$ (conjugate symmetric)

Example: (a) For $u, v \in \mathbb{F}^n$, define:

$$\langle u, v \rangle_{\text{standard}} := \bar{u}_1 \cdot v_1 + \bar{u}_2 \cdot v_2 + \cdots + \bar{u}_n \cdot v_n = u^* v$$

(b) For $u, v \in \mathbb{F}^2$, define:

$$\langle u, v \rangle = \bar{u}_1 \cdot v_2 + \bar{u}_2 \cdot v_1 \rightsquigarrow \text{(2) and (3) satisfied}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = -1 - 1 = -2 < 0 \rightsquigarrow \text{(1) not satisfied}$$

not an inner product!

(c) $\mathcal{P}([0,1], \mathbb{F})$ polynomial space, $p(x) = ix$ is in $\mathcal{P}([0,1], \mathbb{F})$

Define: $$\langle f, g \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx$$

Example: $$\langle p, p \rangle = \int_0^1 \overline{ix} \cdot ix \; dx = \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

$$\left(\sum_{i=1}^n \bar{u}_i v_i \rightsquigarrow \int_0^1 \bar{f} \cdot g \right)$$

Abstract Linear Algebra – Part 11

Example: In $\mathbb{F}^2$:

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \overline{u_1} \cdot v_1 + \overline{u_1} v_2 + \overline{u_2} v_1 + 4 \overline{u_2} v_2$$

$$= \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_{\text{standard}}$$

$\rightsquigarrow$ check 3 rules of inner product

$\hookrightarrow$ $\langle x, x \rangle = \langle x, Ax \rangle_{\text{standard}} > 0$ for $x \neq 0$

Definition: $A \in \mathbb{F}^{n \times n}$ is called a positive definite matrix if:

- $A^* = A$ (selfadjoint/symmetric)
- $\langle x, Ax \rangle_{\text{standard}} > 0$ for all $x \in \mathbb{F}^n \setminus \{0\}$

Fact: If $A \in \mathbb{F}^{n \times n}$ is a positive definite matrix, then

$\langle y, x \rangle := \langle y, Ax \rangle_{\text{standard}}$ defines an inner product in $\mathbb{F}^n$.

Example: $\left\langle x, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} x \right\rangle_{\text{standard}} = \overline{x_1} \cdot x_1 + \overline{x_1} x_2 + \overline{x_2} x_1 + 4 \overline{x_2} x_2$

$$= |x_1 + x_2|^2 + 3 \cdot |x_2|^2 \geq 0$$

If $|x_1 + x_2|^2 + 3 \cdot |x_2|^2 = 0 \Rightarrow |x_1 + x_2|^2 = 0$ and $\underbrace{|x_2|^2 = 0}$

$\Rightarrow x_2 = 0$

$\Rightarrow x_1 = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ positive definite

Proposition: For a selfadjoint matrix $A \in \mathbb{F}^{n \times n}$, the following claims are equivalent:

(a) A positive definite

(b) All eigenvalues of A are positive (> 0)

(c) After Gaussian elimination (without scaling and exchanging rows) only with row operations $Z_{i+\lambda j}$, (see part 37 of Linear Algebra) all pivots in the row echelon form are positive.

(d) The determinants of the so-called leading principal minors of A are positive.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$H_1 = (a_{11})$, $H_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$H_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots, H_n = A$

$$\det(H_1) > 0, \ \det(H_2) > 0, \ldots, \ \det(H_n) > 0$$

(Sylvester's criterion)

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$

(d) $\det(1) = 1 > 0$

$\det \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = 4 - 1 = 3 > 0$

(c) Gaussian elimination: $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{\text{II} - 1\text{I}} \begin{pmatrix} \boxed{1}^{>0} & 1 \\ 0 & \boxed{3}_{>0} \end{pmatrix}$

Abstract Linear Algebra – Part 12

Recall: inner product on the $\mathbb{F}$-vector space V:

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F} \qquad \text{three properties!}$$

For $V = \mathbb{F}^n$: $\langle y, x \rangle = \langle y, Ax \rangle_{\text{standard}}$

positive definite matrix

We use inner products for:

- measuring angles ⟵ Cauchy Schwarz inequality
- measuring lengths: $\|x\| := \sqrt{\langle x, x \rangle}$

norm of x

Cauchy-Schwarz inequality: $\langle \cdot , \cdot \rangle$ inner product on the $\mathbb{F}$-vector space V.

Then: $$|\langle y, x \rangle| \leq \|x\| \cdot \|y\| \qquad \text{for all } x, y \in V$$

and $$|\langle y, x \rangle| = \|x\| \cdot \|y\| \iff x, y \text{ lin. dependent}$$

Proof: (1) For $x = 0$: $\langle y, \underbrace{x}_{0 \cdot v} \rangle = 0 \cdot \langle y, v \rangle = 0$ and $\|x\| \cdot \|y\| = 0$

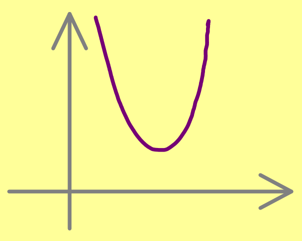
(2) For $x \neq 0$: Show: $\left|\langle y, \underbrace{\tfrac{x}{\|x\|}}_{\hat{x}} \rangle\right| \leq \|y\|$, $\|\hat{x}\| = 1$

For any $\lambda \in \mathbb{R}$: $0 \leq \langle y - \lambda \hat{x}, y - \lambda \hat{x} \rangle$

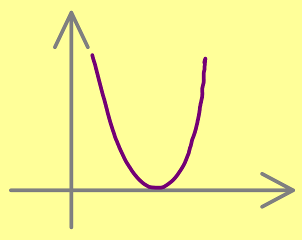
$$= \langle y, y \rangle - \lambda \underbrace{\langle \hat{x}, y \rangle}_{\overline{\alpha}} - \lambda \underbrace{\langle y, \hat{x} \rangle}_{\alpha} + \lambda^2 \langle \hat{x}, \hat{x} \rangle$$

$$= \lambda^2 + \lambda \cdot \underbrace{\left(-2 \cdot \mathrm{Re}(\langle y, \hat{x} \rangle)\right)}_{p} + \underbrace{\|y\|^2}_{q}$$

quadratic polynomial has zeros: $\lambda_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$



or



$$\Longrightarrow \left(\frac{p}{2}\right)^2 - q \leq 0 \quad \Longrightarrow \quad \mathrm{Re}(\langle y, \hat{x} \rangle)^2 \leq \|y\|^2$$

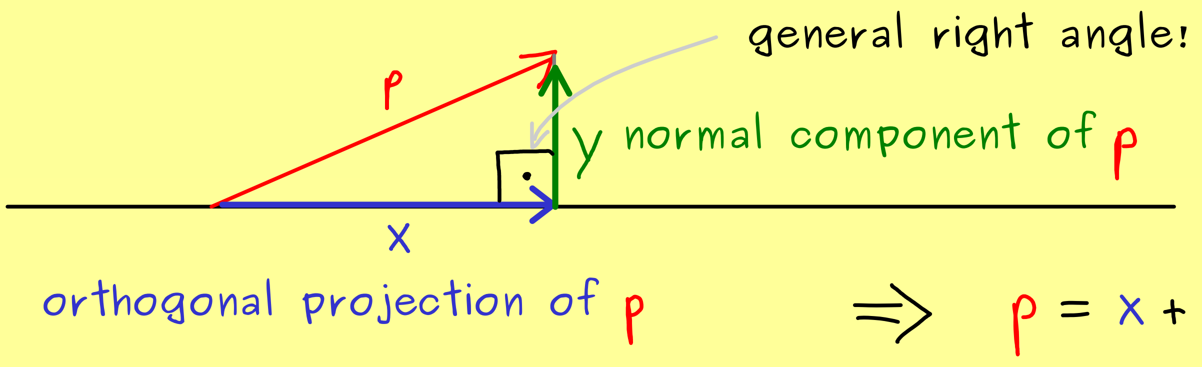
$$\Longrightarrow |\mathrm{Re}(\langle y, \hat{x} \rangle)| \leq \|y\| \qquad \rightsquigarrow \text{Cauchy-Schwarz } \mathbb{F} = \mathbb{R}$$

For $\mathbb{F} = \mathbb{C}$: $\underbrace{e^{i\varphi}}_{c} \langle y, \hat{x} \rangle = |\langle y, \hat{x} \rangle|$

$$\| \qquad\qquad$$

$$|\mathrm{Re}(c \langle y, \hat{x} \rangle)| = |\mathrm{Re}(\langle y, \underbrace{c \hat{x}}_{\tilde{x}} \rangle)| \leq \|y\|$$

Abstract Linear Algebra – Part 13



$\Rightarrow \quad p = x + y$

Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

We say $x, y \in V$ are orthogonal, written as $x \perp y$,

if $\langle x, y \rangle = 0$.

Example: $\mathcal{P}([-1,1], \mathbb{F})$ polynomial space, $\langle f, g \rangle = \int_{-1}^{1} \overline{f(x)}\, g(x)\, dx$

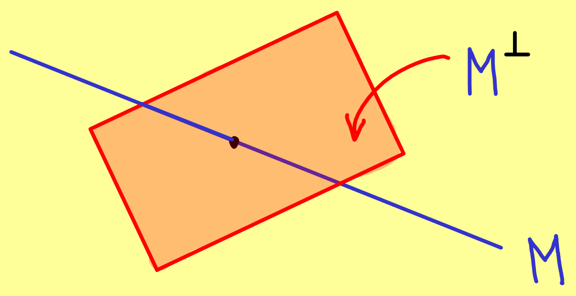
$p_1 : x \mapsto x$
$p_2 : x \mapsto x^2$ $\Rightarrow \langle p_1, p_2 \rangle = \int_{-1}^{1} x^3\, dx = 0 \Rightarrow p_1 \perp p_2$

Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

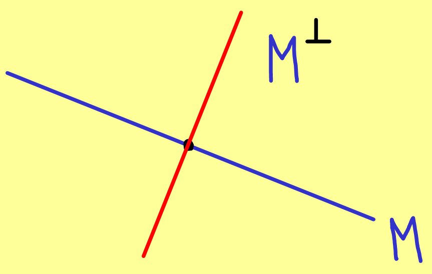
For $M \subseteq V$, $M \neq \emptyset$, we define the orthogonal complement:

$$M^{\perp} := \{ x \in V \mid \langle x, m \rangle = 0 \text{ for all } m \in M \}$$

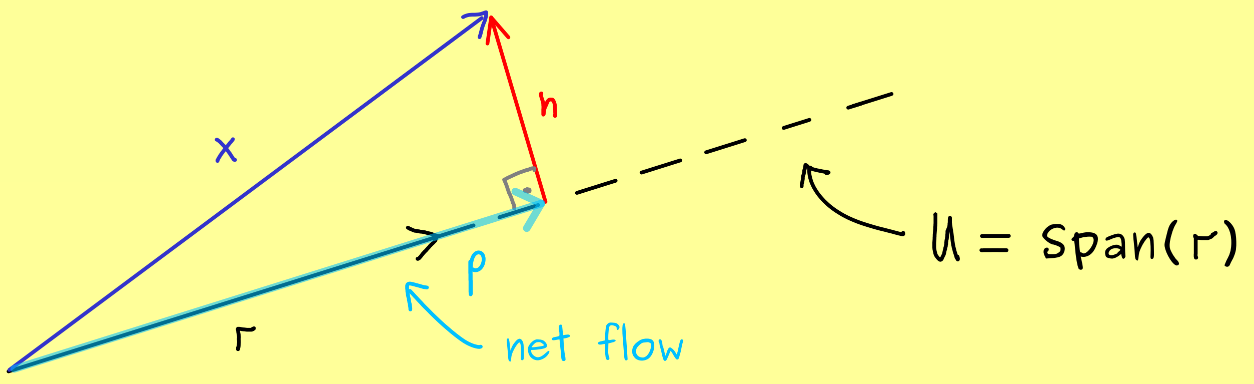
in $\mathbb{R}^3$:



in $\mathbb{R}^2$:



Abstract Linear Algebra – Part 14



Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

Let $U \subseteq V$ be a subspace with $U = \mathrm{Span}(r)$, $r \neq 0$.

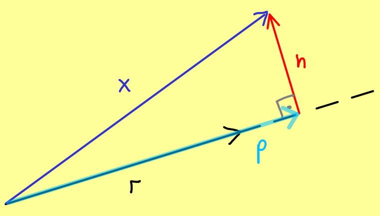
For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \perp r$,

we call:

p orthogonal projection of x onto U

n normal component of x with respect to U

Let's show the uniqueness: Assume $x = \underset{\in U}{p} + \underset{\in U^\perp}{n}$, $x = \underset{\in U}{\tilde{p}} + \tilde{n} \in U^\perp$



$$\Rightarrow \quad p + n = \tilde{p} + \tilde{n} \quad \Rightarrow \quad \underbrace{p - \tilde{p}}_{\in U} = \underbrace{\tilde{n} - n}_{\in U^\perp}$$

$$\Rightarrow \quad 0 = \langle p - \tilde{p}, \tilde{n} - n \rangle = \begin{cases} \langle p - \tilde{p}, p - \tilde{p} \rangle \\ \langle \tilde{n} - n, \tilde{n} - n \rangle \end{cases}$$

$$\overset{\text{inner product is positive definite}}{\Rightarrow} \quad p - \tilde{p} = 0 = \tilde{n} - n \quad \Rightarrow \quad p = \tilde{p} \quad \text{and} \quad n = \tilde{n}$$

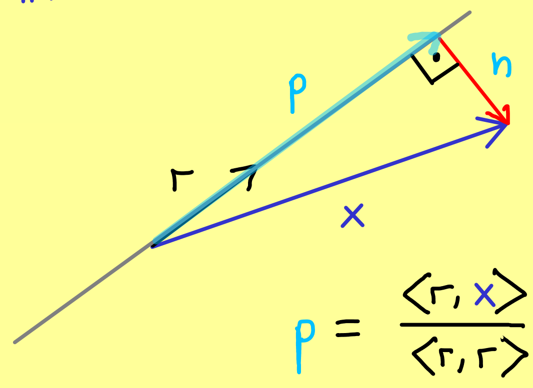
Existence: $p \in U = \mathrm{Span}(r) \Rightarrow p = \lambda \cdot r$ for $\lambda \in \mathbb{F}$

$$\langle r, x \rangle = \langle r, \lambda \cdot r + n \rangle = \lambda \langle r, r \rangle + \underbrace{\langle r, n \rangle}_{= 0}$$

$$\Rightarrow \quad \lambda = \frac{\langle r, x \rangle}{\langle r, r \rangle} \quad \leadsto \quad p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r \quad , \quad n = x - p \quad \checkmark$$

Abstract Linear Algebra – Part 15

Example: $\mathbb{R}^2$ with standard inner product.



$r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\hat{r} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (unit vector)

$x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

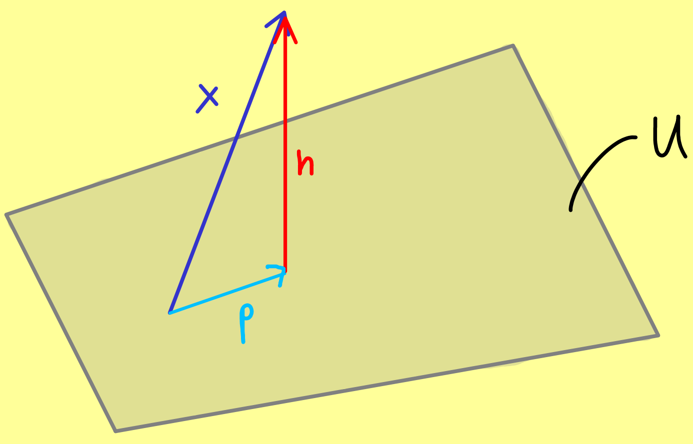
$$p = \frac{\langle r, x \rangle}{\langle r, r \rangle} \cdot r = \langle \hat{r}, x \rangle \cdot \hat{r}$$

(orthogonal projection: $\hat{r} \langle \hat{r}, \cdot \rangle$)

$$\Rightarrow \quad p = \frac{1}{\sqrt{2}}(5+4) \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 9/2 \end{pmatrix}$$

$$\Rightarrow \quad n = x - p = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Generalization:



$$x = \underbrace{p}_{\in U} + n \in U^{\perp}$$

Important fact: $U \cap U^{\perp} = \{0\}$ for every subspace $U \subseteq V$

Proposition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{F}$.

Let $U \subseteq V$ be a k-dimensional subspace, $\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U.

Then for $y \in V$: $\quad y \perp u$ for all $u \in U$

$$\Longleftrightarrow$$

$y \perp b_j$ for all $j \in \{1, 2, \dots, k\}$

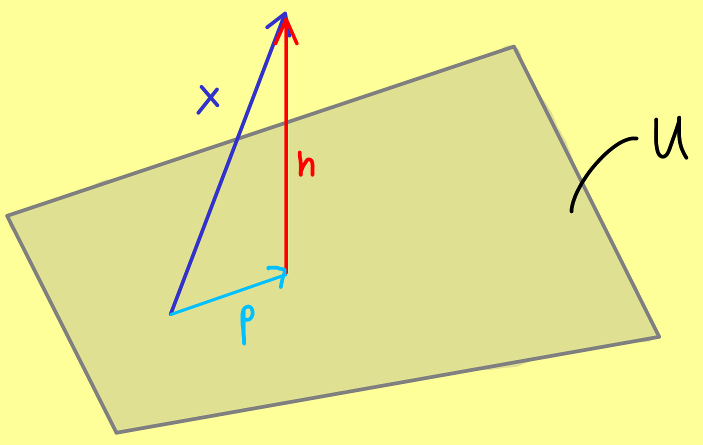
Proof: $(\Rightarrow)$ ✓ $\quad (\Leftarrow)$ We assume: $\langle y, b_j \rangle = 0$ for all $j \in \{1, 2, \dots, k\}$

$$\Rightarrow \sum_{j=1}^{k} \lambda_j \langle y, b_j \rangle = 0$$

$$\Rightarrow \left\langle y, \sum_{j=1}^{k} \lambda_j b_j \right\rangle = 0 \overset{\mathcal{B} \text{ basis}}{\Rightarrow} y \perp u \text{ for all } u \in U$$

Orthogonal projection onto a subspace:

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,
$U \subseteq V$ k-dimensional subspace.



For $x \in V$ and a decomposition $x = p + n$ with $p \in U$, $n \in U^{\perp}$,

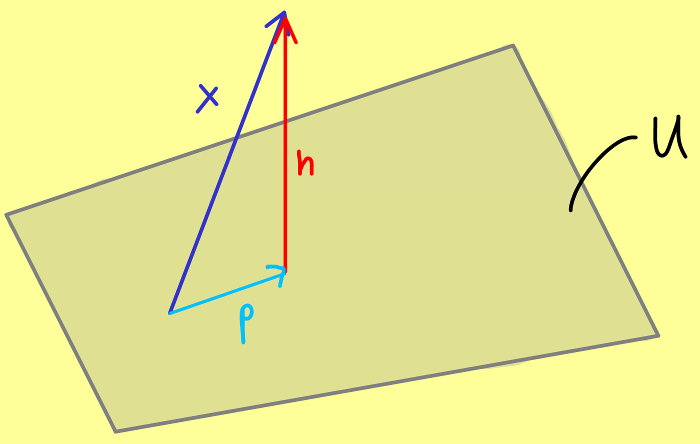
we call:

p	orthogonal projection of x onto U
h	normal component of x with respect to U

The Bright Side of Mathematics

Abstract Linear Algebra – Part 16

Orthogonal projection:



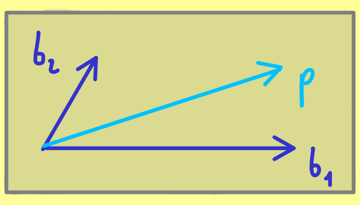
$x = \underbrace{p}_{\in U} + \underbrace{n}_{\in U^\perp}$

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,

$U \subseteq V$ k-dimensional subspace.

Assume we have a basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$ of U.

$p = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_k b_k$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{F}$



For each basis vector b_j: $\langle b_j , x \rangle = \langle b_j , p \rangle + \underbrace{\langle b_j , n \rangle}_{=0}$

$$= \langle b_j , \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_k b_k \rangle$$

$$= \sum_{i=1}^{k} \lambda_i \langle b_j , b_i \rangle$$

Let's rewrite these k linear equations:

$$\underbrace{\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix}}_{\text{Gramian matrix } G(\mathcal{B})} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$\rightsquigarrow$ solution gives us the orthogonal projection

Do we have a unique solution? $G(\mathcal{B})$ invertible $\iff$ $\mathrm{Ker}(G(\mathcal{B})) = \{0\}$

Let's prove $\mathrm{Ker}(G(\mathcal{B})) = \{0\}$: Choose $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \mathrm{Ker}(G(\mathcal{B}))$

$$G(\mathcal{B}) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \implies \text{for all } j: \ \beta_1 \langle b_j , b_1 \rangle + \beta_2 \langle b_j , b_2 \rangle + \cdots + \beta_k \langle b_j , b_k \rangle = 0$$

linearity

$$\implies \text{for all } j: \ \langle b_j , \underbrace{\sum_{i=1}^{k} \beta_i b_i}_{y \in U} \rangle = 0$$

$$\overset{\text{Proposition part 15}}{\implies} y \in U^\perp$$

$$\overset{U \cap U^\perp = \{0\}}{\implies} y = 0 \implies \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example: $\mathbb{R}^3$ with standard inner product, $U = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$$x = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightsquigarrow G(\mathcal{B}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \langle b_2, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix}$$

$$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

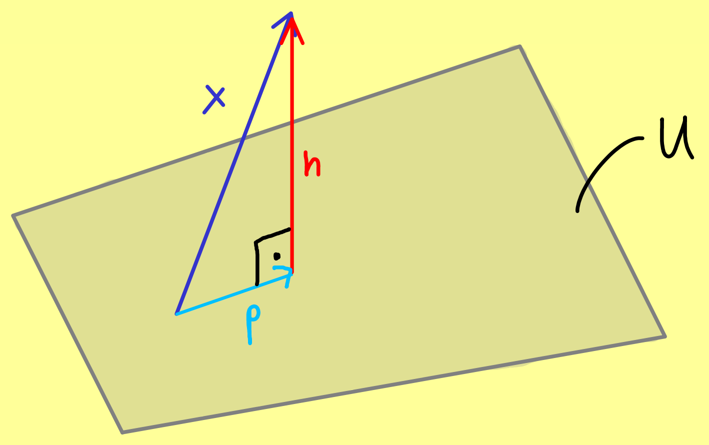
$$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 1 \end{array} \right) \overset{2 \cdot \mathrm{II}}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 2 & 2 & 2 \end{array} \right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & -1 \end{array} \right) \rightsquigarrow \begin{array}{l} \lambda_2 = -1 \\ \lambda_1 = 2 \end{array}$$

$$\implies p = 2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

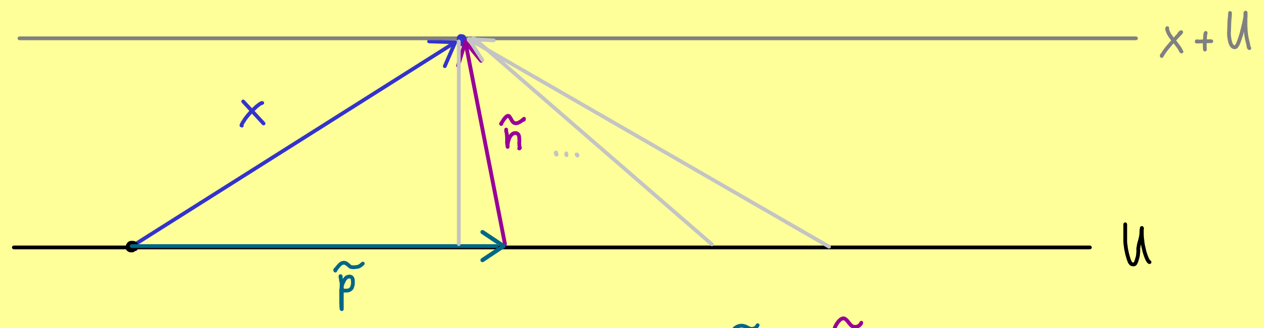
Abstract Linear Algebra – Part 17

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.



$$x = \underset{\in U}{p} + \underset{\in U^\perp}{n}$$
$$= x|_U + x|_{U^\perp}$$

Simplified picture: What is the distance between U and $x + U$?



$$x = \underset{\in U}{\tilde{p}} + \tilde{n}$$

Approximation formula:

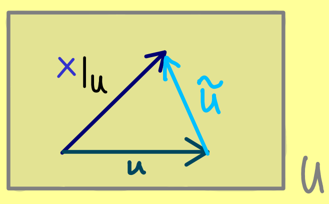
V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

For $x \in V$: $\operatorname{dist}(x, U) := \inf \{ \| x - u \| \mid u \in U \} = \| x - \underbrace{x|_U}_{\text{orthogonal projection}} \|$

> Recall: $\|x\| := \sqrt{\langle x, x \rangle}$ norm of x

Proof: For all $u \in U$: $\| x - u \|^2 = \| \underbrace{(x - x|_U)}_{n} + \underbrace{(x|_U - u)}_{=: \tilde{u} \in U} \|^2$

normal component of x with respect to U



$$= \langle n + \tilde{u} , n + \tilde{u} \rangle$$
$$= \langle n, n \rangle + \underbrace{\langle n, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, n \rangle}_{=0} + \langle \tilde{u}, \tilde{u} \rangle \qquad n \in U^\perp$$
$$= \|n\|^2 + \underbrace{\|\tilde{u}\|^2}_{\geq 0} \geq \|n\|^2$$

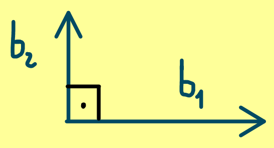
$$\Longrightarrow \inf \{ \| x - u \| \mid u \in U \} \geq \|n\|$$

We have equality $\iff$ $\tilde{u} = 0$ $\iff$ $u = x|_U$ $\square$

Abstract Linear Algebra – Part 18

Assumption: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

Idea: Choose a nice basis $(b_1, b_2, \dots, b_k)$ of U:



$$\langle b_1, b_2 \rangle = 0$$

$$\langle b_1, b_1 \rangle = \|b_1\|^2 = 1 \ , \ \langle b_2, b_2 \rangle = 1$$

Notation: $\langle b_i, b_j \rangle = \delta_{ij} := \begin{cases} 1 \ , & i = j \\ 0 \ , & i \neq j \end{cases}$

Kronecker delta

Orthogonal projection: For $x \in V$: $x = \underbrace{x|_U}_{\in U} + \underbrace{x|_{U^\perp}}_{\in U^\perp}$ can be calculated:

$\mathcal{B} = (b_1, b_2, \dots, b_k)$ basis of U

$$G(\mathcal{B}) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} \langle b_1, x \rangle \\ \vdots \\ \langle b_k, x \rangle \end{pmatrix} \rightsquigarrow \text{solving LES gives } x|_U$$

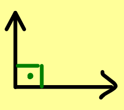
Gramian matrix

$$G(\mathcal{B}) = \begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_k \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_k \rangle \\ \vdots & & & \\ \langle b_k, b_1 \rangle & \langle b_k, b_2 \rangle & \cdots & \langle b_k, b_k \rangle \end{pmatrix} \overset{\substack{\text{nice} \\ \text{basis}}}{=} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\text{identity matrix}}$$

$$\Rightarrow \quad x|_U = \sum_{j=1}^{k} b_j \langle b_j, x \rangle$$

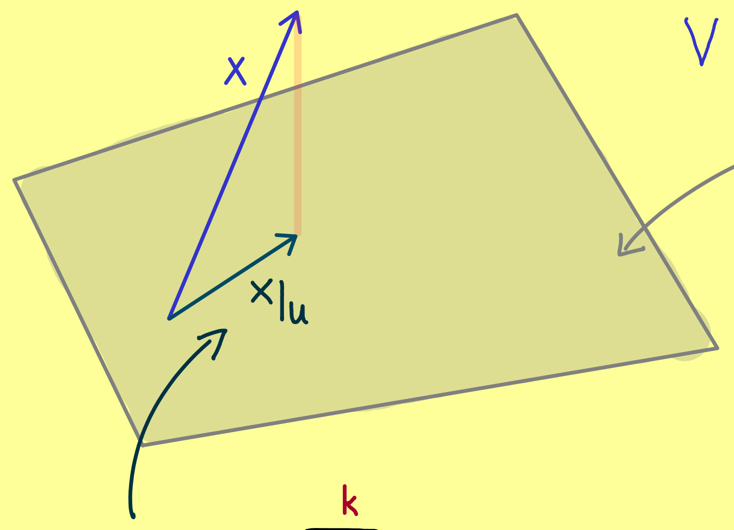
Definition: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

A family $(b_1, b_2, \dots, b_m)$ (with $b_j \in U$) is called:

- orthogonal system (OS) if $\langle b_i, b_j \rangle = 0$ for all $i \neq j$ 
- orthonormal system (ONS) if $\langle b_i, b_j \rangle = \delta_{ij}$
- orthogonal basis (OB) if it's an OS and a basis of U
- orthonormal basis (ONB) if it's an ONS and a basis of U

Example: $\mathbb{R}^3$ with standard inner product, $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$ ONB of $\mathbb{R}^3$.

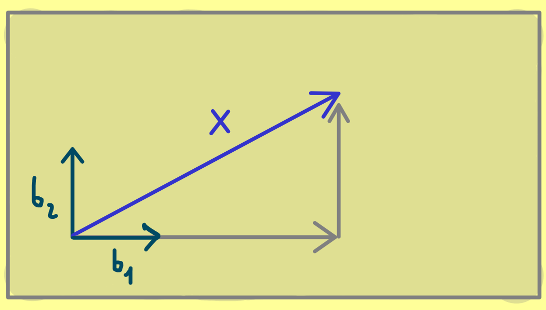
Abstract Linear Algebra – Part 19



V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$,
$U \subseteq V$ k-dimensional subspace,
$\mathcal{B} = (b_1, b_2, \dots, b_k)$ ONB of U.

orthogonal projection: $x|_U = \sum_{j=1}^{k} b_j \underbrace{\langle b_j , x \rangle}_{\text{scalars}}$

The case $x \in U$:



$x = \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_k b_k$

How to find?

↳ easy for ONB!

Result: V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ONB of U.

Then for each $u \in U$ we have the linear combination

$$u = \sum_{j=1}^{k} b_j \underbrace{\langle b_j , u \rangle}_{\in \mathbb{F}}$$

(Fourier expansion of u w.r.t. $\mathcal{B}$)

Fourier coefficients

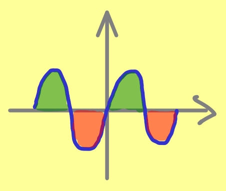
Example: $V = U = \mathrm{Span}\left(x \mapsto \frac{1}{\sqrt{2}} , x \mapsto \cos(x) , x \mapsto \cos(2x) , x \mapsto \sin(x) \right)$

(subspace in $\mathcal{F}(\mathbb{R})$)

with inner product: $\langle f , g \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) g(x) \, dx$

We get: $\langle x \mapsto \cos(x) , x \mapsto \cos(x) \rangle = \frac{1}{\pi} \cdot \underbrace{\int_{-\pi}^{\pi} (\cos(x))^2 \, dx}_{\pi} = 1$

$\langle x \mapsto \cos(x) , x \mapsto \sin(x) \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \underbrace{\cos(x) \sin(x)}_{\text{odd function}} \, dx$



$= 0$

$\vdots$

$\Longrightarrow$ $\mathcal{B} = \left(x \mapsto \frac{1}{\sqrt{2}} , x \mapsto \cos(x) , x \mapsto \cos(2x) , x \mapsto \sin(x) \right)$ ONB

Take u with $u(x) = (\sin(x))^2$ (actually $u \in V$)

Calculate: $\langle b_1 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} (\sin(x))^2 \, dx = \frac{1}{\sqrt{2}}$

$\langle b_2 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \cos(x) (\sin(x))^2 \, dx = \frac{1}{\pi} \, \frac{1}{3} (\sin(x))^3 \Big|_{-\pi}^{\pi} = 0$

$\langle b_3 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \cos(2x) (\sin(x))^2 \, dx = -\frac{1}{2}$

longer calculation

$\langle b_4 , u \rangle = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (\sin(x))^3 \, dx = 0$

$\Longrightarrow$ $u = b_1 \langle b_1 , u \rangle + b_3 \langle b_3 , u \rangle$

$$(\sin(x))^2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \cos(2x) \cdot \left(-\frac{1}{2}\right) = \frac{1}{2} \cdot \left(1 - \cos(2x)\right)$$

Abstract Linear Algebra – Part 20

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

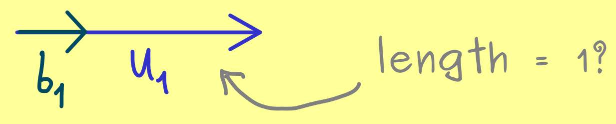
basis of U: $(u_1, u_2, \dots, u_k)$ $\rightsquigarrow$ ONB of U: $(b_1, b_2, \dots, b_k)$

Gram-Schmidt process/algorithm

$$\langle b_i , b_j \rangle = \delta_{ij}$$

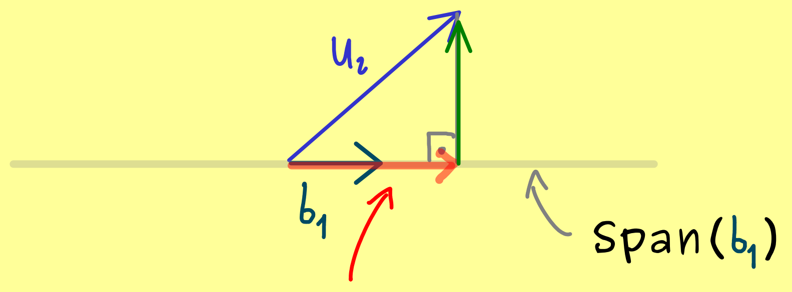
Gram-Schmidt orthonormalization:

(1) Normalize first vector:



$$b_1 := \frac{1}{\|u_1\|} \cdot u_1 \quad \text{where} \quad \|u_1\| := \sqrt{\langle u_1 , u_1 \rangle}$$

(2) Next vector u_2:

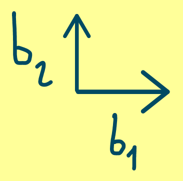


orthogonal projection of u_2 onto Span(b_1):

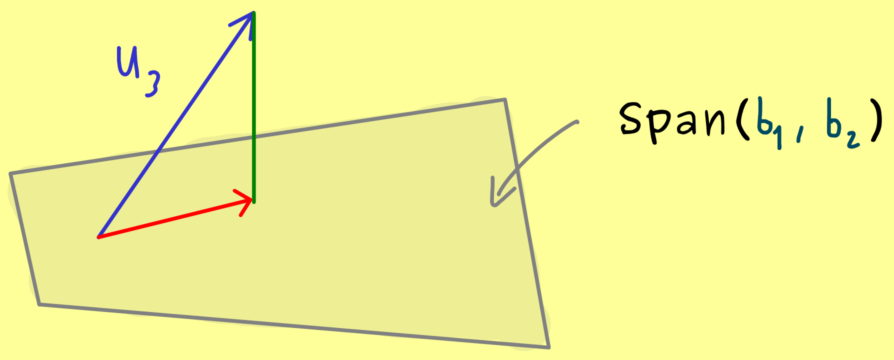
$$\hookrightarrow u_2 \big|_{\mathrm{Span}(b_1)} = b_1 \langle b_1 , u_2 \rangle$$

normal component: $\tilde{b}_2 = u_2 - b_1 \langle b_1 , u_2 \rangle$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$



(3) Next vector u_3:



orthogonal projection of u_3 onto Span(b_1, b_2):

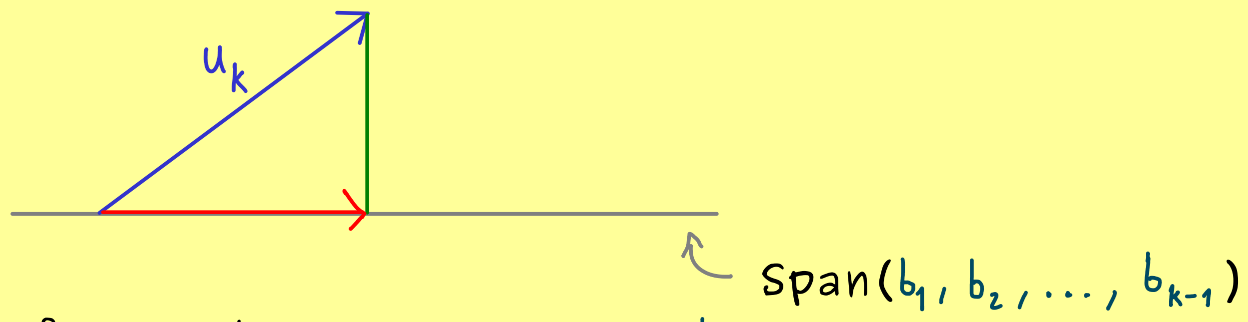
$$\hookrightarrow u_3 \big|_{\mathrm{Span}(b_1, b_2)} := b_1 \langle b_1 , u_3 \rangle + b_2 \langle b_2 , u_3 \rangle$$

normal component: $\tilde{b}_3 = u_3 - b_1 \langle b_1 , u_3 \rangle - b_2 \langle b_2 , u_3 \rangle$

normalize it: $b_3 := \frac{1}{\|\tilde{b}_3\|} \tilde{b}_3$

⋮ continue!

(k) Next vector u_k:



orthogonal projection of u_k onto Span$(b_1, b_2, \dots, b_{k-1})$

$$\hookrightarrow u_k \big|_{\mathrm{Span}(b_1, b_2, \dots, b_{k-1})} := \sum_{j=1}^{k-1} b_j \langle b_j , u_k \rangle$$

normal component: $\tilde{b}_k = u_k - \sum_{j=1}^{k-1} b_j \langle b_j , u_k \rangle$

normalize it: $b_k := \frac{1}{\|\tilde{b}_k\|} \tilde{b}_k \implies$ ONB of U: $(b_1, b_2, \dots, b_k)$

The Bright Side of Mathematics

Abstract Linear Algebra – Part 21

V $\mathbb{F}$-vector space, inner product $\langle \cdot , \cdot \rangle$, $U \subseteq V$ k-dimensional subspace.

basis of U: $(u_1, u_2, \dots, u_k)$ $\rightsquigarrow$ ONB of U: $(b_1, b_2, \dots, b_k)$

Gram–Schmidt process/algorithm

Example: $V = \mathcal{P}([-1,1], \mathbb{R})$ polynomial space with inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Take $U = \mathcal{P}_2([-1,1], \mathbb{R})$ with basis (m_0, m_1, m_2)

(polynomials of degree ≤ 2)   not ONB!

$m_0: x \mapsto 1$
$m_1: x \mapsto x$
$m_2: x \mapsto x^2$

Gram–Schmidt orthonormalization:

(1) Normalize first vector: $\|m_0\|^2 = \langle m_0, m_0 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = 2$

$$b_0 := \frac{1}{\|m_0\|} \cdot m_0 = \frac{1}{\sqrt{2}} m_0 , \qquad b_0(x) = \frac{1}{\sqrt{2}}$$

(2) Next vector m_1:

normal component: $\tilde{b}_1 = m_1 - b_0 \underbrace{\langle b_0, m_1 \rangle}_{= \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot x \, dx = 0} = m_1$

normalize it: $b_1 := \frac{1}{\|\tilde{b}_1\|} \tilde{b}_1$, $\|\tilde{b}_1\|^2 = \int_{-1}^{1} x \cdot x \, dx = \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{2}{3}$

$$= \sqrt{\tfrac{3}{2}}\, m_1 , \qquad b_1(x) = \sqrt{\tfrac{3}{2}}\, x$$

(3) Next vector m_2:

normal component: $\tilde{b}_2 = m_2 - b_0 \underbrace{\langle b_0, m_2 \rangle}_{= \int_{-1}^{1} \frac{1}{\sqrt{2}} \cdot x^2 \, dx = \frac{1}{\sqrt{2}} \cdot \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3}} - b_1 \underbrace{\langle b_1, m_2 \rangle}_{= \int_{-1}^{1} \sqrt{\frac{3}{2}}\, x \cdot x^2 \, dx = 0}$

$$= m_2 - \frac{1}{3} m_0 , \qquad \tilde{b}_2(x) = x^2 - \frac{1}{3}$$

normalize it: $b_2 := \frac{1}{\|\tilde{b}_2\|} \tilde{b}_2$, $\|\tilde{b}_2\|^2 = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right) dx$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3} x^2 + \frac{1}{9}\right) dx$$

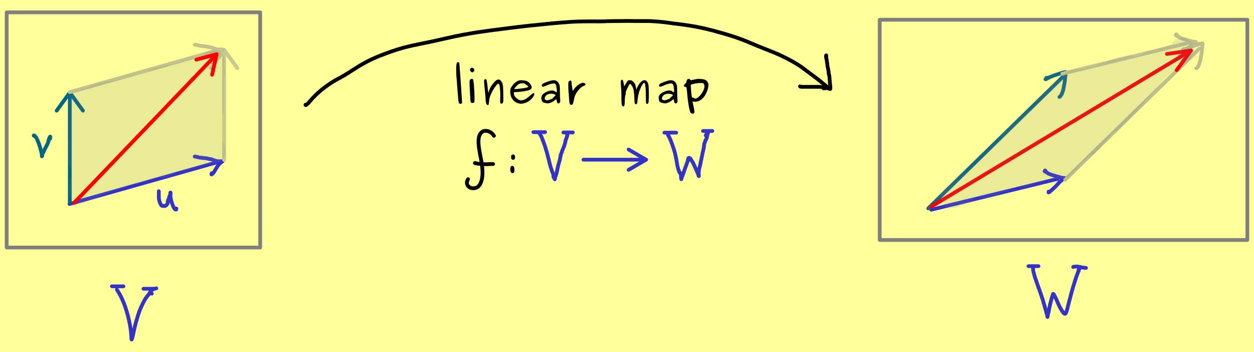
$$= \frac{1}{5} x^5 - \frac{2}{9} x^3 + \frac{1}{9} x \Big|_{-1}^{1} = \frac{8}{45}$$

$$\Rightarrow \quad b_2(x) = \sqrt{\tfrac{45}{8}} \cdot \left(x^2 - \frac{1}{3}\right)$$

$\rightsquigarrow$ ONB for $\mathcal{P}_2([-1,1], \mathbb{R})$

(Legendre polynomials)

Abstract Linear Algebra – Part 22



Recall: $f: \mathbb{R}^n \to \mathbb{R}^m$ linear $\iff$ matrix $A \in \mathbb{R}^{m \times n}$

Definition: Let V, W be two $\mathbb{F}$-vector spaces. (same $\mathbb{F}$ for both)

A map $f: V \to W$ is called linear if:

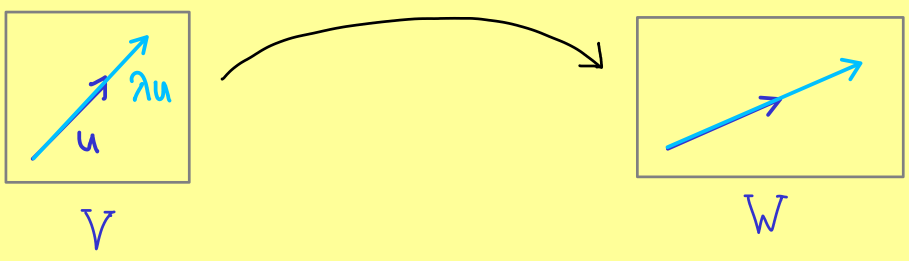
(1) $f(u + v) = f(u) + f(v)$

vector addition in V vector addition in W

(2) $f(\lambda \cdot u) = \lambda \cdot f(u)$

scalar multiplication in V scalar multiplication in W

for all $u, v \in V$, $\lambda \in \mathbb{F}$.



Remember: $f(0_V) = f(0 \cdot u) \overset{(2)}{=} 0 \cdot f(u) = 0_W$

Example: (a) $V = \mathbb{F}^3$, $W = \mathbb{F}$, $a \in V$.

$f(u) := \langle a, u \rangle_{\text{standard}}$ is a linear map.

$\quad\quad = a^* u$ (matrix multiplication)

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^* = (\overline{a_1} \;\; \overline{a_2} \;\; \overline{a_3})$$

(transpose + complex conjugation)

(b) $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$

$\ell : V \to W$
$\quad p \mapsto p'$

is a linear map!

$\ell(x \mapsto x^2) = x \mapsto 2x$

$\ell(p + q) = (p + q)' = p' + q' = \ell(p) + \ell(q)$

$\ell(\lambda p) = (\lambda p)' = \lambda p' = \lambda \ell(p)$

BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Abstract Linear Algebra – Part 23

Recall: linear map or linear operator $\ell: V \longrightarrow W$:

$$\ell(x+y) = \ell(x) + \ell(y)$$
$$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$$

Definition: Let V, W be two $\mathbb{F}$-vector spaces. (same $\mathbb{F}$ for both)

For $k: V \longrightarrow W$, $\ell: V \longrightarrow W$ linear maps, we define:

$$k+\ell : V \longrightarrow W, \quad (k+\ell)(x) := k(x) + \ell(x)$$

(given $\lambda \in \mathbb{F}$) $$\lambda \cdot \ell : V \longrightarrow W, \quad (\lambda \cdot \ell)(x) := \lambda \cdot \ell(x)$$

$$\begin{tikzcd} \boxed{V} \arrow[r, bend left, "\ell"] & \boxed{W} \end{tikzcd}$$

$$\begin{tikzcd} \boxed{V} \arrow[r, bend left, "\lambda\cdot\ell"] & \boxed{W} \end{tikzcd}$$

Result: With $+, \cdot$ from above, the set $\mathcal{L}(V,W) = \{ \ell: V \longrightarrow W \mid \text{linear} \}$ forms an $\mathbb{F}$-vector space.

Zero vector $o \in \mathcal{L}(V,W)$ is given by the zero map $o(x) = o_W$ for all $x \in V$.

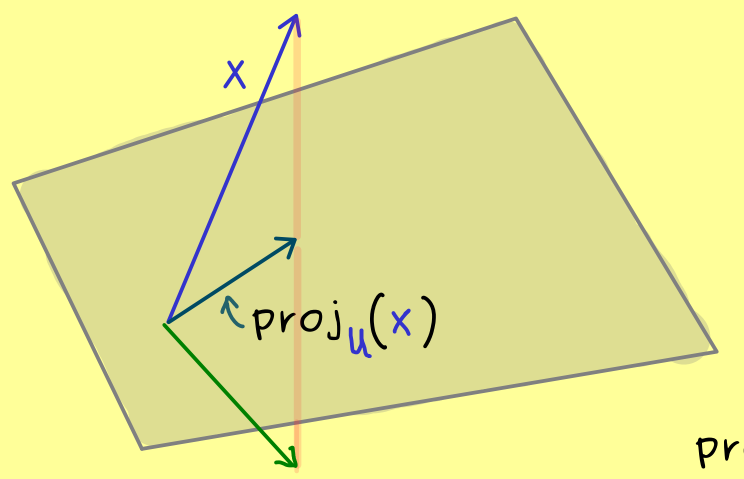
zero vector in W

Example: V with inner product $\langle \cdot , \cdot \rangle$ and ONB $(e_1, e_2, \dots, e_n)$.

$U = \mathrm{Span}(e_1, e_2, \dots, e_{n-1})$

Orthogonal projection onto U: $\mathrm{proj}_U : V \longrightarrow V$ linear map

$$x \longmapsto \sum_{j=1}^{n-1} e_j \langle e_j, x \rangle$$



$\mathrm{proj}_{U^\perp} : V \longrightarrow V$ linear map

$$x \longmapsto e_n \langle e_n, x \rangle$$

Addition: $\mathrm{proj}_U + \mathrm{proj}_{U^\perp} = \mathrm{id}_V$

Subtraction: $\mathrm{proj}_U - \mathrm{proj}_{U^\perp} = \mathrm{id}_V - 2 \cdot \mathrm{proj}_{U^\perp}$ reflection

Composition:

$$\begin{tikzcd} \boxed{U} \arrow[r, bend left, "k \text{ linear}"] \arrow[rr, bend right, "\ell \circ k \text{ linear}"'] & \boxed{V} \arrow[r, bend left, "\ell \text{ linear}"] & \boxed{W} \end{tikzcd}$$

$$k \in \mathcal{L}(U,V), \ \ell \in \mathcal{L}(V,W) \implies \ell \circ k \in \mathcal{L}(U,W)$$

Example: $\mathrm{proj}_U \circ \mathrm{proj}_U = \mathrm{proj}_U$, $\mathrm{proj}_U \circ \mathrm{proj}_{U^\perp} = o$

zero vector in $\mathcal{L}(V,V)$

Abstract Linear Algebra – Part 24

$\ell : V \longrightarrow W$ linear map preserves the structure of the vector space.

$\parallel$

(vector space) homomorphism

$$\begin{tikzcd} V \arrow[r, bend left, "\ell"] \arrow[rr, bend right, "k\circ\ell"'] & W \arrow[r, bend left, "k"] & V \end{tikzcd}$$

Reminder: (just maps on sets) $f : V \longrightarrow W$ is called invertible if there is a map $g : W \longrightarrow V$ with $g \circ f = \mathrm{id}_V$ and $f \circ g = \mathrm{id}_W$

denoted by f^{-1}

f bijective $\Longleftrightarrow$ f invertible

Fact: $\ell : V \longrightarrow W$ linear + bijective $\Rightarrow$ $\ell^{-1} : W \longrightarrow V$ linear

(see part 31 in "Linear Algebra")

Example:

$$\begin{tikzcd} \boxed{\text{abstract vector space } V \quad \text{fix basis } \mathcal{B}} \arrow[d, bend left, "\Phi_{\mathcal{B}} \text{ linear}"] \\ \boxed{\text{concrete vector space } \mathbb{F}^n \quad \text{standard basis}} \arrow[u, bend left, "\Phi_{\mathcal{B}}^{-1} \text{ linear}"] \end{tikzcd}$$

$\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V

$\Phi_{\mathcal{B}}(b_j) = e_j$

$\boxed{\text{basis isomorphism}}$

$(e_1, e_2, \dots, e_n)$ standard basis of $\mathbb{F}^n$

Definition: $\ell : V \longrightarrow W$ homomorphism + $\ell^{-1} : W \longrightarrow V$ homomorphism

$\rightsquigarrow$ is called an isomorphism

Remember: (vector space) isomorphism $=$ bijective linear map

$\parallel$

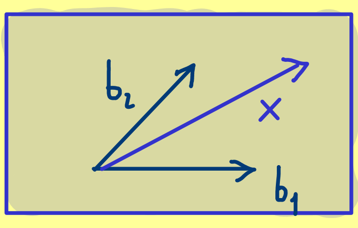
linear isomorphism

Abstract Linear Algebra – Part 25

$\ell: V \to W$ linear: $\ell(x+y) = \ell(x) + \ell(y)$

$\ell(\lambda \cdot x) = \lambda \cdot \ell(x)$

V



$\mathcal{B} = (b_1, b_2, \dots, b_n)$ basis of V

$\Longrightarrow$ $x \in V$ can be written as $x = \alpha_1 b_1 + \dots + \alpha_n b_n$

Hence: $\ell(x) = \ell(\alpha_1 b_1 + \dots + \alpha_n b_n)$

$= \alpha_1 \ell(b_1) + \alpha_2 \ell(b_2) + \dots + \alpha_n \ell(b_n)$

If know these, we know ℓ

Example: $V = \mathcal{P}_3(\mathbb{R})$, $W = \mathcal{P}_2(\mathbb{R})$, $\ell: V \to W$, $p \mapsto p'$

is a linear map!

basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$\ell(m_0) = 0$ ← zero vector: $x \mapsto 0$

$\ell(m_k) = k \cdot m_{k-1}$, $k \in \{1,2,3\}$

Result:

$$\begin{tikzcd}
V \text{ with basis: } \mathcal{B} = (b_1,\dots,b_n) \arrow[rr, bend left, "\ell \text{ linear}"] \arrow[d, bend left, "\Phi_{\mathcal{B}}"] & & W \text{ with basis: } \mathcal{C} = (c_1,\dots,c_m) \arrow[d, "\Phi_{\mathcal{C}}"] \\
\mathbb{F}^n \text{ (standard basis)} \arrow[u, bend left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[rr, bend left, "f \text{ linear}"] & & \mathbb{F}^m \text{ (standard basis)}
\end{tikzcd}$$

$f = \Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}$

linear $f: \mathbb{F}^n \to \mathbb{F}^m$

↳ can be represented by a matrix

$f(x) = Ax$ ← $\in \mathbb{F}^{m \times n}$

First column of A: $f(e_1) = \left(\Phi_{\mathcal{C}} \circ \ell \circ \Phi_{\mathcal{B}}^{-1}\right)(e_1) = \Phi_{\mathcal{C}}(\ell(b_1))$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Matrix representation: For a linear map $\ell: V \to W$,

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} := \begin{pmatrix} | & | & & | \\ \Phi_{\mathcal{C}}(\ell(b_1)) & \Phi_{\mathcal{C}}(\ell(b_2)) & \cdots & \Phi_{\mathcal{C}}(\ell(b_n)) \\ | & | & & | \end{pmatrix} \in \mathbb{F}^{m \times n}$$

is called the matrix representation of ℓ with respect to $\mathcal{B}$ and $\mathcal{C}$.

Example (from before)

$V = \mathcal{P}_3(\mathbb{R})$ basis: $\mathcal{B} = (b_1, b_2, b_3, b_4) = (m_0, m_1, m_2, m_3)$

with $m_0: x \mapsto 1$, $m_k: x \mapsto x^k$

$\ell: V \to W$, $p \mapsto p'$

is a linear map!

$W = \mathcal{P}_2(\mathbb{R})$ basis: $\mathcal{C} = (c_1, c_2, c_3) = (m_0, m_1, m_2)$

$\Phi_{\mathcal{C}}(\ell(b_1)) = \Phi_{\mathcal{C}}(\ell(m_0)) = \Phi_{\mathcal{C}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$

$\Phi_{\mathcal{C}}(\ell(b_2)) = \Phi_{\mathcal{C}}(\ell(m_1)) = \Phi_{\mathcal{C}}(m_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{F}^3$

$\vdots$

$$\Longrightarrow \quad \ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$ matrix representation of ℓ

Abstract Linear Algebra - Part 26

Matrix representations:

$$\begin{tikzcd}[column sep=huge, row sep=large]
V \ \text{with basis: } \mathcal{B} = (b_1,\dots,b_n) \arrow[r, "\ell \text{ linear}"] \arrow[d, shift left, "\Phi_{\mathcal{B}}"] \arrow[rr, bend left, "k \circ \ell"] & W \ \text{with basis: } \mathcal{C} = (c_1,\dots,c_m) \arrow[r, "k \text{ linear}"] \arrow[d, shift left, "\Phi_{\mathcal{C}}"] & U \ \text{with basis: } \mathcal{D} = (d_1,\dots,d_k) \arrow[d, shift left, "\Phi_{\mathcal{D}}"] \\
\mathbb{F}^n \ \text{(standard basis)} \arrow[u, shift left, "\Phi_{\mathcal{B}}^{-1}"] \arrow[r, "\ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}"'] & \mathbb{F}^m \ \text{(standard basis)} \arrow[u, shift left, "\Phi_{\mathcal{C}}^{-1}"] \arrow[r, "k_{\mathcal{D} \leftarrow \mathcal{C}} \in \mathbb{F}^{k \times m}"'] & \mathbb{F}^k \ \text{(standard basis)} \arrow[u, shift left, "\Phi_{\mathcal{D}}^{-1}"]
\end{tikzcd}$$

We get: $(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}}$ (matrix product)

Example: $\ell : \mathbb{R}^3 \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto (v_1 + v_2 + v_3) \cdot m_0 + (v_1 + v_2) \cdot m_1 + v_1 \cdot m_2$

with $m_0 : x \mapsto 1$, $m_k : x \mapsto x^k$

$k : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}^2$, $p \mapsto \begin{pmatrix} p'(1) \\ p(1) - p''(1) \end{pmatrix}$

Picture:

$$\begin{tikzcd}
\mathbb{R}^3, \ \mathcal{B} \arrow[r, bend left, "\ell"] & \mathcal{P}_2, \ \mathcal{C} \arrow[r, bend left, "k"] & \mathbb{R}^2, \ \mathcal{D}
\end{tikzcd}$$

standard basis $\qquad (m_0, m_1, m_2) \qquad$ standard basis

$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = \, ?$

$$\ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} | & | & | \\ \Phi_{\mathcal{C}}(\ell(b_1)) & \Phi_{\mathcal{C}}(\ell(b_2)) & \Phi_{\mathcal{C}}(\ell(b_3)) \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$k_{\mathcal{D} \leftarrow \mathcal{C}} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

$$(k \circ \ell)_{\mathcal{D} \leftarrow \mathcal{B}} = k_{\mathcal{D} \leftarrow \mathcal{C}} \, \ell_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

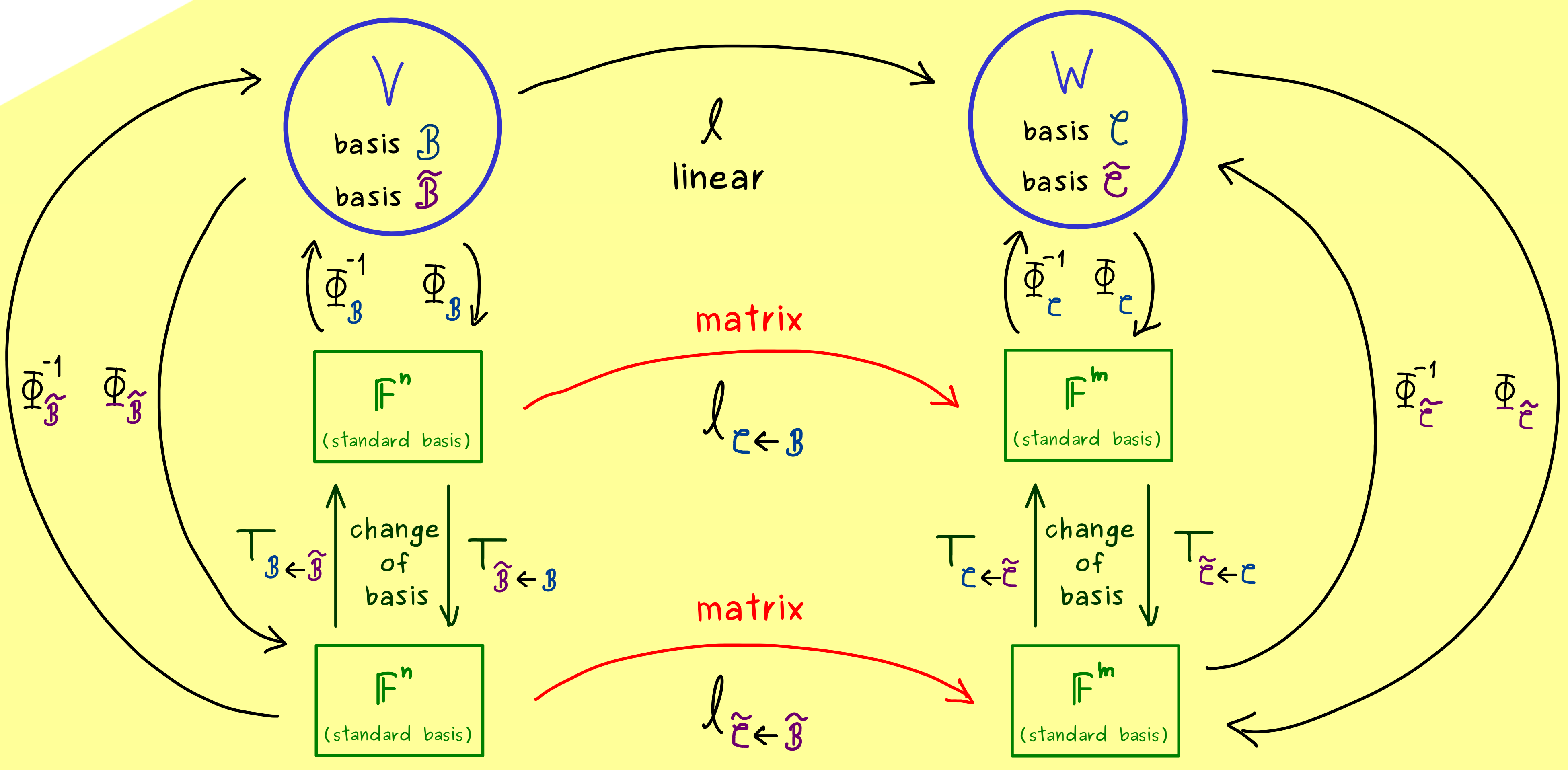
Corollary: $(\ell^{-1})_{\mathcal{B} \leftarrow \mathcal{C}} = (\ell_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}$

$n = m$

$$\begin{tikzcd}[column sep=huge]
V \ \text{with basis: } \mathcal{B} = (b_1,\dots,b_n) \arrow[r, bend left, "\ell \text{ linear}"] & W \ \text{with basis: } \mathcal{C} = (c_1,\dots,c_m) \arrow[l, bend left, "\ell^{-1}"]
\end{tikzcd}$$

The Bright Side of Mathematics

Abstract Linear Algebra - Part 27



Result: $\boxed{\ell_{\widetilde{\mathcal{C}}\leftarrow\widehat{\mathcal{B}}} = T_{\widetilde{\mathcal{C}}\leftarrow\mathcal{C}}\,\ell_{\mathcal{C}\leftarrow\mathcal{B}}\,T_{\mathcal{B}\leftarrow\widehat{\mathcal{B}}}}$

Example: $\ell: \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\ell(p) = p'$ linear map!

$\mathcal{B} = (m_3, m_2, m_1, m_0)$ $\quad$ $\mathcal{C} = (m_2, m_1, m_0)$

$\widehat{\mathcal{B}} = (2m_3 - m_1,\ m_2 + m_0,\ m_1 + m_0,\ m_1 - m_0)$, $\widetilde{\mathcal{C}} = (m_2 - \frac{1}{2}m_1,\ m_2 + \frac{1}{2}m_1,\ m_0)$

matrix representation: $\ell_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

change-of-basis matrices: $T_{\mathcal{B}\leftarrow\widehat{\mathcal{B}}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$

$T_{\mathcal{C}\leftarrow\widetilde{\mathcal{C}}} = \begin{pmatrix} 1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{inverse}} T_{\widetilde{\mathcal{C}}\leftarrow\mathcal{C}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\ell_{\widetilde{\mathcal{C}}\leftarrow\widehat{\mathcal{B}}} = T_{\widetilde{\mathcal{C}}\leftarrow\mathcal{C}}\,\ell_{\mathcal{C}\leftarrow\mathcal{B}}\,T_{\mathcal{B}\leftarrow\widehat{\mathcal{B}}} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

Abstract Linear Algebra - Part 28

Fact: $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 3 & -2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$ are different but

they describe the same linear map $\ell : \mathcal{P}_3(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$, $\ell(p) = p'$

with respect to different bases.

Question: $\ell : V \longrightarrow W$ linear, $A = \ell_{\mathcal{C} \leftarrow \mathcal{B}} \in \mathbb{F}^{m \times n}$.

For another $\widetilde{A} \in \mathbb{F}^{m \times n}$, can we find bases such that $\widetilde{A} = \ell_{\widetilde{\mathcal{C}} \leftarrow \widetilde{\mathcal{B}}}$ **?**

If YES!, then we say A and $\widetilde{A}$ are equivalent.

Picture:

$$\begin{tikzcd}
\boxed{\mathcal{B}}\ \mathbb{F}^n \text{ (standard basis)} \arrow[rr, "\ell_{\mathcal{C} \leftarrow \mathcal{B}}"] \arrow[d, shift left, "T_{\widetilde{\mathcal{B}} \leftarrow \mathcal{B}}"] & & \mathbb{F}^m \text{ (standard basis)}\ \boxed{\mathcal{C}} \arrow[d, shift left, "T_{\widetilde{\mathcal{C}} \leftarrow \mathcal{C}}"] \\
\boxed{\widetilde{\mathcal{B}}}\ \mathbb{F}^n \text{ (standard basis)} \arrow[rr, "\ell_{\widetilde{\mathcal{C}} \leftarrow \widetilde{\mathcal{B}}}"] \arrow[u, shift left, "T_{\mathcal{B} \leftarrow \widetilde{\mathcal{B}}}"] & & \mathbb{F}^m \text{ (standard basis)}\ \boxed{\widetilde{\mathcal{C}}} \arrow[u, shift left, "T_{\mathcal{C} \leftarrow \widetilde{\mathcal{C}}}"]
\end{tikzcd}$$

change of basis — change of basis

invertible! — invertible!

Definition: A matrix $\widetilde{A} \in \mathbb{F}^{m \times n}$ is called equivalent to a matrix $A \in \mathbb{F}^{m \times n}$

if there are invertible matrices $S \in \mathbb{F}^{m \times m}$, $T \in \mathbb{F}^{n \times n}$, such that:

$$\widetilde{A} = S\,A\,T .$$

We write: $\widetilde{A} \sim A$

Remark: $\sim$ defines an equivalence relation on $\mathbb{F}^{m \times n}$:

(1) reflexive: $A \sim A$ for all $A \in \mathbb{F}^{m \times n}$

(2) symmetric: $A \sim B \Rightarrow B \sim A$ for all $A, B \in \mathbb{F}^{m \times n}$

(3) transitive: $A \sim B \wedge B \sim C \Rightarrow A \sim C$ for all $A, B, C \in \mathbb{F}^{m \times n}$

Abstract Linear Algebra - Part 29

Equivalence relation: $A, B \in \mathbb{F}^{m \times n}$, $A \sim B$ ← they both represent the same linear map $\ell : V \longrightarrow W$

← there are invertible matrices S, T with $B = SAT$.

kernel and range?

$$\mathrm{Ker}(B) = \mathrm{Ker}(SAT) = \{ x \in \mathbb{F}^n \mid A\underbrace{Tx}_{\in \mathrm{Ker}(A)} = 0 \} = T^{-1}\mathrm{Ker}(A)$$

$$\mathrm{Ran}(B) = \mathrm{Ran}(SAT) = \{ SATx \mid x \in \mathbb{F}^n \}$$

$$= \{ S\underbrace{A\tilde{x}}_{\in \mathrm{Ran}(A)} \mid \tilde{x} \in \mathbb{F}^n \} = S\,\mathrm{Ran}(A)$$

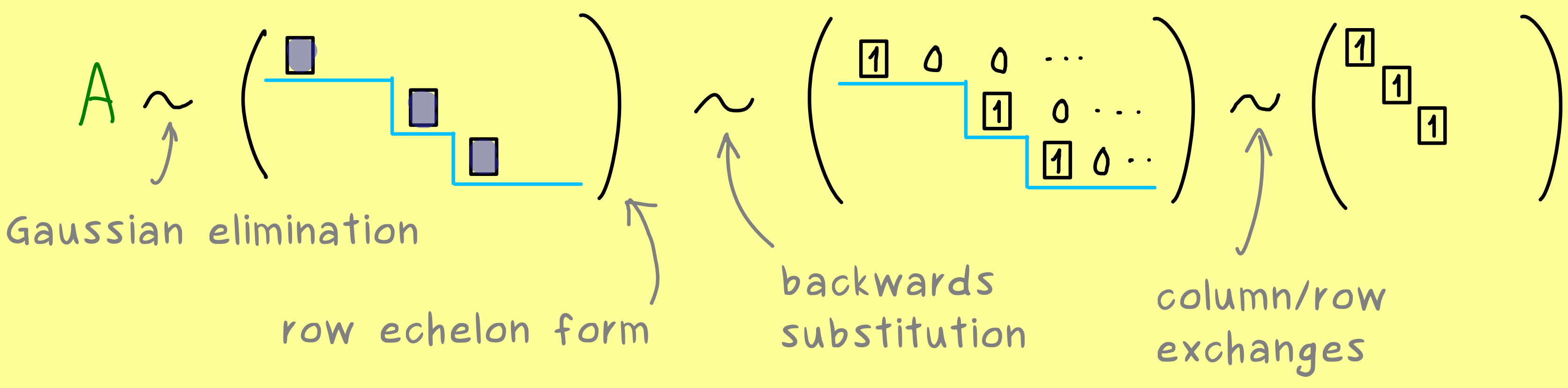
Result: $A \sim B \implies \mathrm{rank}(A) = \mathrm{rank}(B)$

$$\begin{matrix} \mathrm{rank}(A) & = & \mathrm{rank}(B) \\ + & & + \\ \mathrm{nullity}(A) & = & \mathrm{nullity}(B) \\ \| & & \| \\ n & & n \end{matrix}$$

Proposition: For $A, B \in \mathbb{F}^{m \times n}$, we have:

$$A \sim B \iff \mathrm{rank}(A) = \mathrm{rank}(B)$$

Proof:

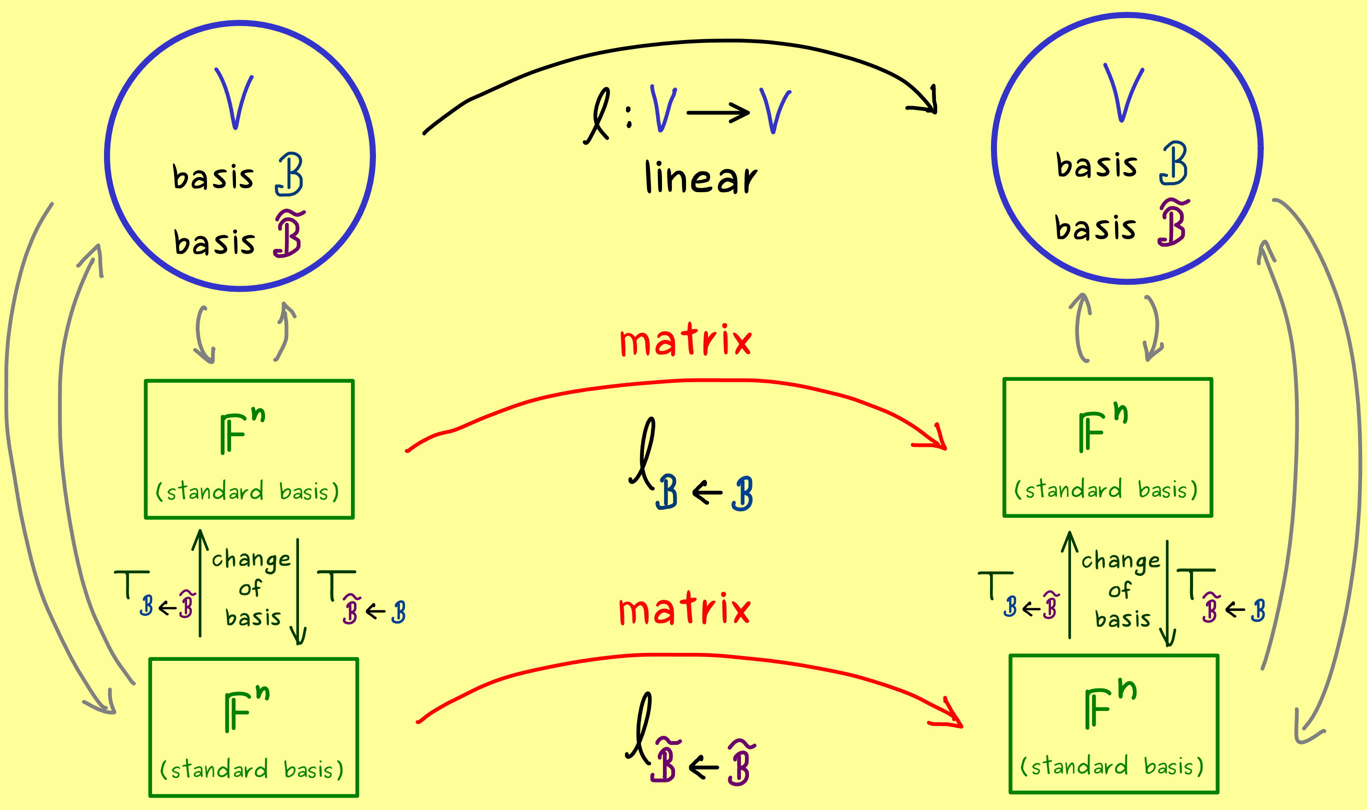


$$\implies A \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad r = \mathrm{rank}(A)$$

$$B \sim \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} \quad\quad r = \mathrm{rank}(B)$$

$\square$

Abstract Linear Algebra - Part 30



We have:

$$\ell_{\widehat{\mathcal{B}} \leftarrow \widehat{\mathcal{B}}} = T_{\widehat{\mathcal{B}} \leftarrow \mathcal{B}} \; \ell_{\mathcal{B} \leftarrow \mathcal{B}} \; T_{\mathcal{B} \leftarrow \widehat{\mathcal{B}}}$$

$$\tilde{A} = T^{-1} A \, T$$

Definition: A matrix $\tilde{A} \in \mathbb{F}^{n \times n}$ is called similar to a matrix $A \in \mathbb{F}^{n \times n}$ if there is an invertible $T \in \mathbb{F}^{n \times n}$ such that:

$$\tilde{A} = T^{-1} A \, T .$$

We write: $\tilde{A} \approx A$.

Remark: $\approx$ defines an equivalence relation on $\mathbb{F}^{n \times n}$:

(1) reflexive: $A \approx A$ for all $A \in \mathbb{F}^{n \times n}$

(2) symmetric: $A \approx B \Rightarrow B \approx A$ for all $A, B \in \mathbb{F}^{n \times n}$

(3) transitive: $A \approx B \wedge B \approx C \Rightarrow A \approx C$ for all $A, B, C \in \mathbb{F}^{n \times n}$

Easy to see: $A \approx B \Rightarrow A \sim B$

Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \not\approx \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$T^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\approx$ is characterized by the so-called Jordan normal form

Abstract Linear Algebra - Part 31

$\ell: V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces (finite-dimensional).

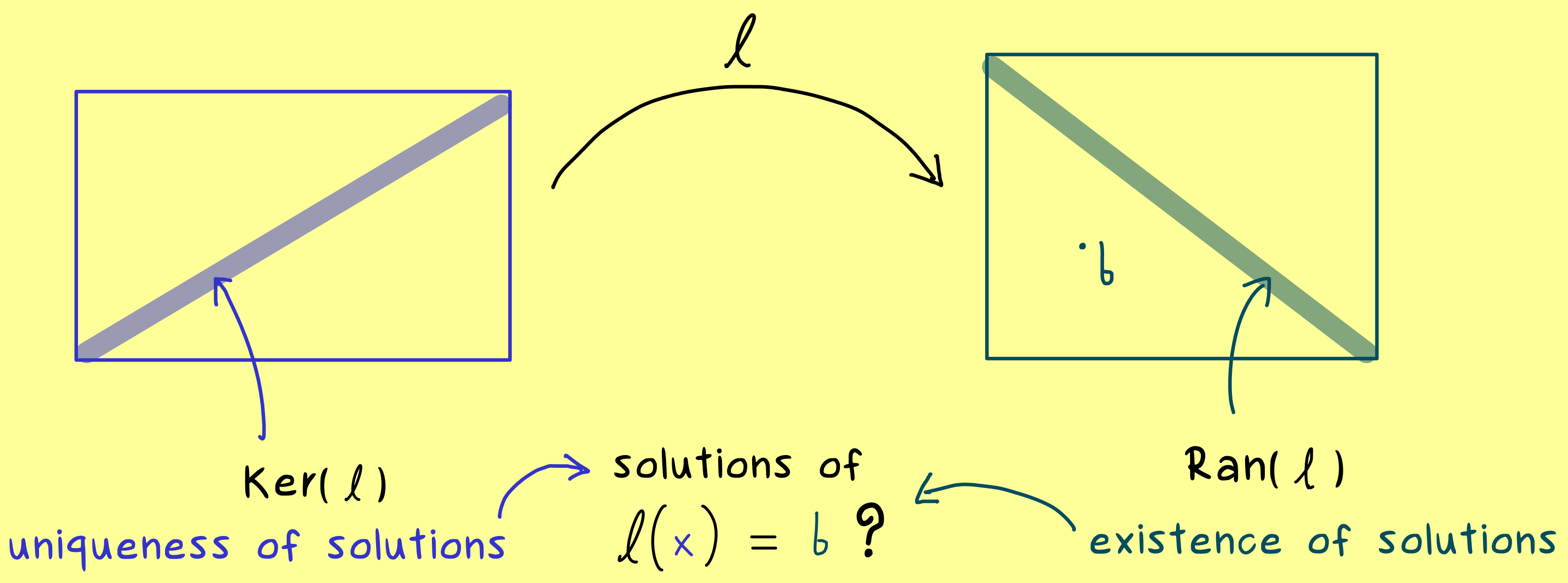
For $b \in W$: $\ell(x) = b$ solutions $x \in V$

matrix representation

$\ell_{\mathcal{C} \leftarrow \mathcal{B}} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ (system of linear equations)

<u>Definition:</u> $\mathrm{Ker}(\ell) := \{ x \in V \mid \ell(x) = 0 \}$ <u>kernel</u> of the linear map ℓ

$\mathrm{Ran}(\ell) := \{ w \in W \mid \text{there is } x \in V \text{ with } \ell(x) = w \}$ <u>range</u> of ℓ



<u>Proposition:</u> $\ell: V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces, $b \in W$.

The solution set $\mathcal{S} := \{ x \in V \mid \ell(x) = b \}$

is either empty or an affine subspace: $\mathcal{S} = \emptyset$ or

$\mathcal{S} = x_0 + \mathrm{Ker}(\ell)$ (with $x_0 \in V$)

<u>Proof:</u> Assume $x_0 \in \mathcal{S}$ $(\ell(x_0) = b)$.

Take any $v \in V$ and look at $x_0 + v$:

$$x_0 + v \in \mathcal{S} \iff \ell(x_0 + v) = b \overset{\text{linear map}}{\iff} \underbrace{\ell(x_0)}_{= b} + \ell(v) = b$$

$$\iff \ell(v) = 0 \iff v \in \mathrm{Ker}(\ell) \qquad \square$$

<u>Rank-nullity theorem:</u> $\ell: V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces (finite-dimensional)

$$\dim(\mathrm{Ran}(\ell)) + \dim(\mathrm{Ker}(\ell)) = \dim(V)$$

|| part 28/29 || ||

with matrix representations

$$\dim(\mathrm{Ran}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})) + \dim(\mathrm{Ker}(\ell_{\mathcal{C} \leftarrow \mathcal{B}})) = n$$

The Bright Side of Mathematics

Abstract Linear Algebra - Part 32

$\ell : V \longrightarrow W$ linear, V, W $\mathbb{F}$-vector spaces

$$\dim(\mathrm{Ran}(\ell)) + \dim(\mathrm{Ker}(\ell)) = \dim(V)$$

$\rightsquigarrow$ helps for solving linear equation $\ell(x) = b$

Example: $V = W = \mathcal{P}_3(\mathbb{R})$ together with monomial basis $(m_3, m_2, m_1, m_0) =: \mathcal{B}$

with $m_0 : x \mapsto 1$, $m_k : x \mapsto x^k$

$$\begin{aligned} \ell : V &\to W \\ p &\mapsto p' \end{aligned} \quad \Longrightarrow \quad \ell(m_k) = k \cdot m_{k-1}, \quad \ell(m_0) = 0$$

matrix representation: $\ell_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$\mathrm{Ker}(\ell_{\mathcal{B}\leftarrow\mathcal{B}}) = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$\mathrm{Ran}(\ell_{\mathcal{B}\leftarrow\mathcal{B}}) = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

Recall general picture: $\ell = \Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B}\leftarrow\mathcal{B}} \circ \Phi_{\mathcal{B}}$

$$\begin{tikzcd} V \arrow[r, "\ell"] \arrow[d, "\Phi_{\mathcal{B}}"'] & W \arrow[d, "\Phi_{\mathcal{B}}"] \\ \mathbb{R}^4 \arrow[r, "\ell_{\mathcal{B}\leftarrow\mathcal{B}}"] & \mathbb{R}^4 \end{tikzcd}$$

$$\begin{aligned} \mathrm{Ker}(\ell) &= \mathrm{Ker}(\Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B}\leftarrow\mathcal{B}} \circ \Phi_{\mathcal{B}}) \\ &= \Phi_{\mathcal{B}}^{-1}\, \mathrm{Ker}(\ell_{\mathcal{B}\leftarrow\mathcal{B}}) = \Phi_{\mathcal{B}}^{-1}\, \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \mathrm{Span}(m_0) \end{aligned}$$

$$\begin{aligned} \mathrm{Ran}(\ell) &= \mathrm{Ran}(\Phi_{\mathcal{B}}^{-1} \circ \ell_{\mathcal{B}\leftarrow\mathcal{B}} \circ \Phi_{\mathcal{B}}) \\ &= \Phi_{\mathcal{B}}^{-1}\, \mathrm{Ran}(\ell_{\mathcal{B}\leftarrow\mathcal{B}}) = \Phi_{\mathcal{B}}^{-1}\, \mathrm{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \mathrm{Span}(m_2, m_1, m_0) \end{aligned}$$

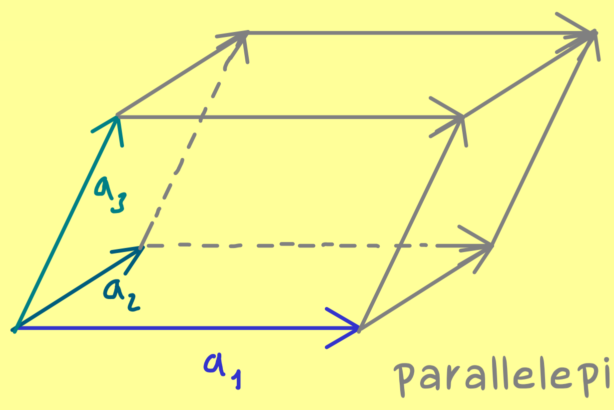
Linear equation: $\ell(p) = g$?

$\rightsquigarrow$ solutions give antiderivatives/primitives for g

$$\Longrightarrow \quad \mathcal{S} = \emptyset \quad \text{or} \quad \mathcal{S} = \tilde{p} + \mathrm{Ker}(\ell) \quad \text{with} \quad \tilde{p}' = g$$

Abstract Linear Algebra - Part 33

Determinant function: det : $\underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$



parallelepiped

$\rightsquigarrow \det(A) \in \mathbb{C}$ for $A \in \mathbb{C}^{n \times n}$

For a linear map $f_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $x \mapsto Ax$

represented by $A \in \mathbb{C}^{n\times n}$, define:

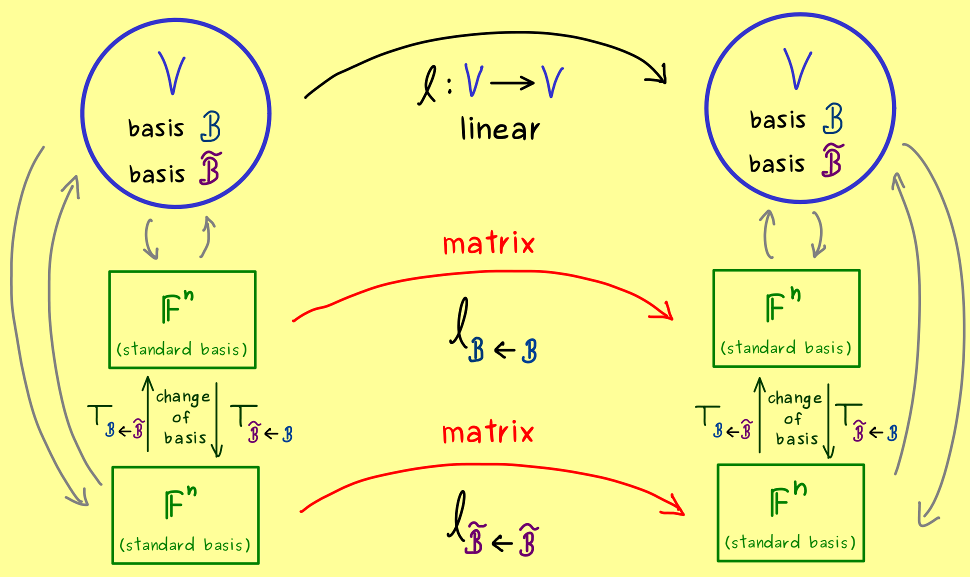
$$\det(f_A) := \det(A)$$

Definition: V $\mathbb{F}$-vector space, $\ell : V \longrightarrow V$ linear, $\mathcal{B} = (b_1, \dots, b_n)$ basis of V.

Then the matrix representation $\ell_{\mathcal{B} \leftarrow \mathcal{B}} \in \mathbb{F}^{n\times n}$ is a square matrix

and we can set: $\det(\ell) := \det(\ell_{\mathcal{B}\leftarrow\mathcal{B}})$.

What happens if we take another basis $\widetilde{\mathcal{B}}$?



$$\ell_{\widetilde{\mathcal{B}}\leftarrow\widetilde{\mathcal{B}}} = T_{\widetilde{\mathcal{B}}\leftarrow\mathcal{B}} \, \ell_{\mathcal{B}\leftarrow\mathcal{B}} \left(T_{\widetilde{\mathcal{B}}\leftarrow\mathcal{B}}\right)^{-1}$$

similar matrices

$$\det(TAT^{-1}) = \det(T)\det(A)\det(T)^{-1}$$
$$= \det(A)$$

$$\Longrightarrow \det(\ell_{\widetilde{\mathcal{B}}\leftarrow\widetilde{\mathcal{B}}}) = \det(\ell_{\mathcal{B}\leftarrow\mathcal{B}})$$

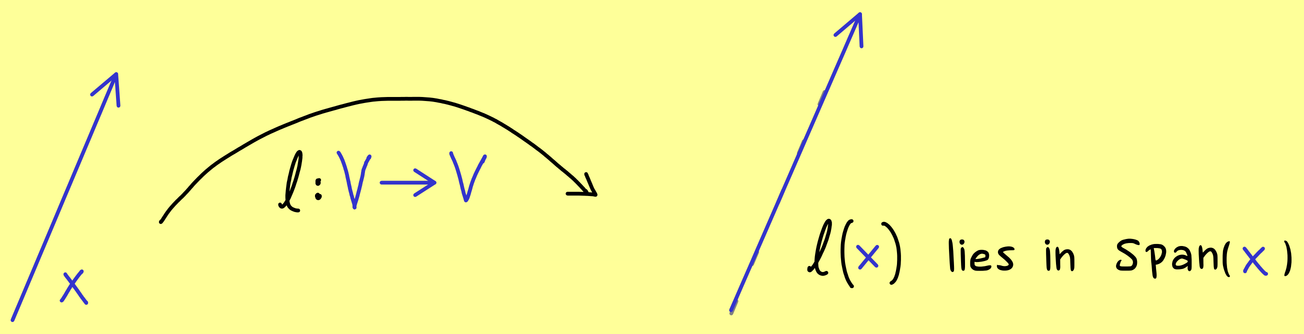
Properties: $\det(\ell \circ k) = \det(\ell) \cdot \det(k)$

$\det(\mathrm{id}) = 1$

$\det(\ell^{-1}) = \det(\ell)^{-1}$

Abstract Linear Algebra - Part 34

eigenvector



Definition: Let V be an $\mathbb{F}$-vector space and $\ell : V \longrightarrow V$ be linear.

A vector $x \in V \setminus \{0\}$ is called an eigenvector of ℓ if

$$\exists \lambda \in \mathbb{F} \ : \quad \ell(x) = \lambda \cdot x$$

λ: eigenvalue of ℓ associated to x

Remember: If $\operatorname{Ker}(\ell - \lambda \cdot \mathrm{id}) \neq \{0\}$, then λ is an eigenvalue of ℓ.

$\underbrace{\operatorname{Ker}(\ell - \lambda \cdot \mathrm{id})}_{\text{eigenspace of } \ell \text{ associated to } \lambda} \setminus \{0\}$ — set of all eigenvectors of ℓ associated to λ

For the finite dimensional case: Let $\mathcal{B}$ be a basis V.

Then: $(\ell - \lambda \cdot \mathrm{id})_{\mathcal{B} \leftarrow \mathcal{B}} = \ell_{\mathcal{B} \leftarrow \mathcal{B}} - \lambda \cdot \mathbb{1}$

Hence: $\operatorname{Ker}(\ell - \lambda \cdot \mathrm{id}) \neq \{0\} \iff \operatorname{Ker}(\ell_{\mathcal{B} \leftarrow \mathcal{B}} - \lambda \cdot \mathbb{1}) \neq \{0\}$

λ eigenvalue of $\ell \iff \lambda$ eigenvalue of $\ell_{\mathcal{B} \leftarrow \mathcal{B}}$

$\det(\ell - \lambda \cdot \mathrm{id}) = 0 \iff \det(\ell_{\mathcal{B} \leftarrow \mathcal{B}} - \lambda \cdot \mathbb{1}) = 0$

Example: $V = C^\infty(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is arbitrarily often continuously differentiable} \}$

$\ell : V \longrightarrow V, \quad f \mapsto f'$ linear map

$\boxed{\exp: x \mapsto e^x}$

$\ell(\exp) = \exp$

eigenvalue: 1

eigenvector = eigenfunction