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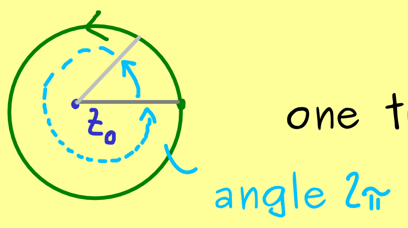
ON STEADY

The Bright Side of Mathematics

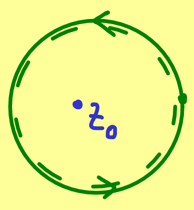
Complex Analysis – Part 24

Winding number for curves

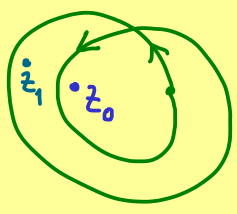
$\gamma : [a,b] \to \mathbb{C}$ piecewise continuously differentiable



one turn around z_0



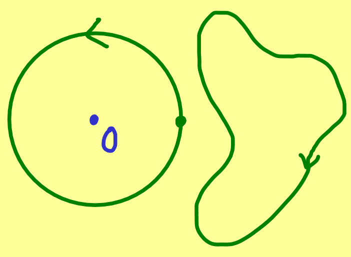
two turns around z_0



two turns around z_0

one turn around z_1

Special integral:



$$\oint_\gamma \frac{1}{z}\,dz = \begin{cases} 2\pi i, & \text{for } \gamma : [0,2\pi] \to \mathbb{C},\ t \mapsto e^{it} \\ 4\pi i, & \text{for } \gamma : [0,4\pi] \to \mathbb{C},\ t \mapsto e^{it} \\ 0, & \text{for } \gamma : [a,b] \to \mathbb{C} \text{ with image in a disk where } 0 \notin \text{disk} \end{cases}$$

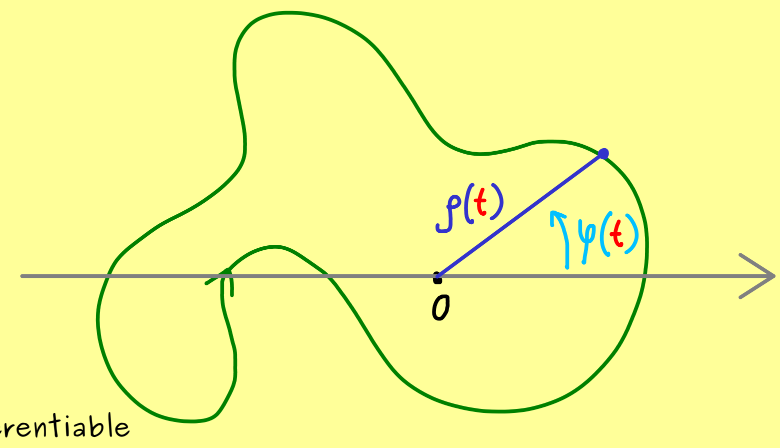
$$\Rightarrow \quad \frac{1}{2\pi i}\oint_\gamma \frac{1}{z}\,dz = 1 \quad \text{for } \gamma : [0,2\pi] \to \mathbb{C},\ t \mapsto e^{it}$$

Definition: The winding number of a curve γ around $z_0 \in \mathbb{C}$ $\left(z_0 \notin \mathrm{Ran}(\gamma)\right)$ is defined by:

$$\mathrm{wind}(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz$$

Fact: γ closed $\Rightarrow$ $\mathrm{wind}(\gamma, z_0) \in \mathbb{Z}$

Proof: Assume $z_0 = 0$, $\gamma : [a,b] \to \mathbb{C}$ closed



Write γ as: $\gamma(t) = \rho(t)\cdot e^{i\varphi(t)}$

piecewise continuously differentiable

$$\oint_\gamma \frac{1}{z}\,dz = \int_a^b \frac{1}{\gamma(t)}\gamma'(t)\,dt = \int_a^b \frac{1}{\rho(t)\cdot e^{i\varphi(t)}}\left(\rho'(t)\cdot e^{i\varphi(t)} + \rho(t)\, i\varphi'(t)\, e^{i\varphi(t)}\right)dt$$

$$= \int_a^b \frac{\rho'(t)}{\rho(t)}\,dt + i\int_a^b \varphi'(t)\,dt$$

$$= \log\left(\rho(t)\right)\Big|_a^b + i\,\varphi(t)\Big|_a^b$$

$$= 0 + i\,2\pi k$$

γ closed
$\Rightarrow \rho(b) = \rho(a)$
$\varphi(b) = \varphi(a) + 2\pi k$
$k \in \mathbb{Z}$

$$\Rightarrow \quad \mathrm{wind}(\gamma, 0) = k$$