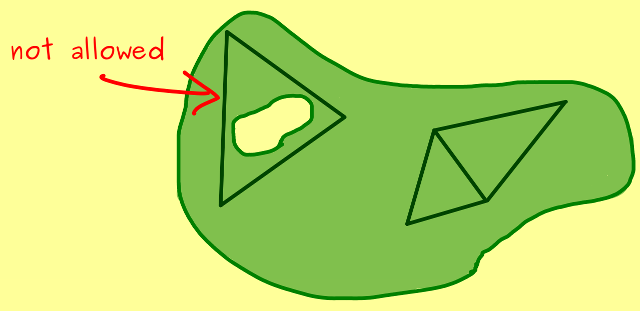
BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

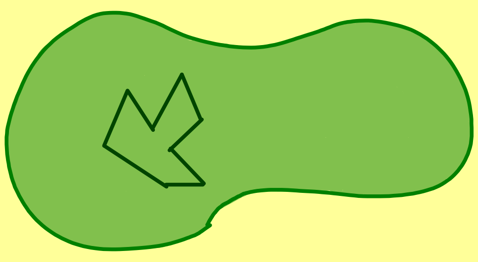
Complex Analysis – Part 23

$f : D \to \mathbb{C}$ holomorphic



$$\triangle \subseteq D \quad \Rightarrow \quad \oint_{\triangle} f(z)\, dz = 0$$

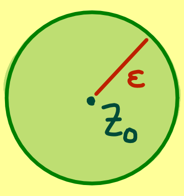
$$\text{▲} \subseteq D \quad \Rightarrow \quad \oint f(z)\, dz = 0$$



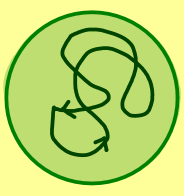
$$\text{▲} \subseteq D \quad \Rightarrow \quad \oint f(z)\, dz = 0$$

Cauchy's theorem (for a disc):

$f : D \to \mathbb{C}$ holomorphic where $D = B_\varepsilon(z_0)$ (open disc)



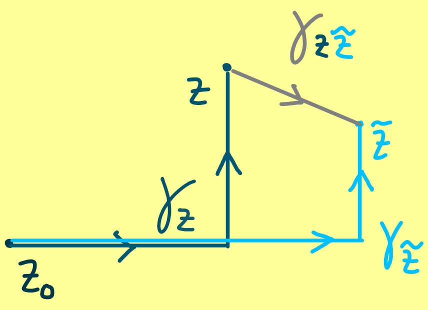
$\gamma : [a,b] \to D$ closed curve



$$\Rightarrow \oint_\gamma f(z)\, dz = 0$$

Proof: Show that an antiderivative exists!

$$F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta$$



(everything lies in D)

$$\oint_{\gamma_{\tilde z} + \gamma_z^- + \gamma_{z\tilde z}^-} f(z)\, dz \overset{\text{Goursat}}{=} 0 \quad (*)$$

$$\left| \frac{F(\tilde z) - F(z)}{\tilde z - z} - f(z) \right| = \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{\tilde z}} f(\zeta)\, d\zeta - \int_{\gamma_z} f(\zeta)\, d\zeta - f(z)(\tilde z - z) \right|$$

$$\overset{(*)}{=} \frac{1}{|\tilde z - z|} \cdot \left| \int_{\gamma_{z\tilde z}} \big(f(\zeta) - f(z) \big)\, d\zeta \right|$$

$$\leq \frac{1}{\cancel{|\tilde z - z|}} \max_{\zeta \in \operatorname{Ran}(\gamma_{z\tilde z})} \big| f(\zeta) - f(z) \big| \cdot \cancel{\text{length}}(\gamma_{z\tilde z}) \xrightarrow{\tilde z \to z} 0$$

$\Rightarrow$ f has an antiderivative on D $\Rightarrow$ $\oint_\gamma f(z)\, dz = 0$ for each closed curve γ in D