The Bright Side of Mathematics

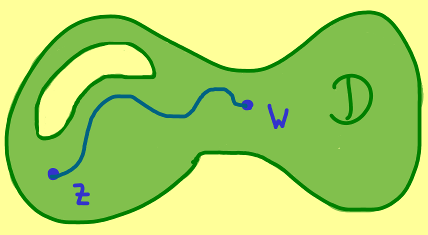
Complex Analysis – Part 21

Fact: $f: U \to \mathbb{C}$ has an antiderivative $\overset{?}{\Longleftrightarrow}$ $\oint_\gamma f(z)\,dz = 0$ for all closed curves γ

Theorem: $\underset{\smile}{f: D \to \mathbb{C}}$ holomorphic

open domain/region: $\underbrace{\text{open} + \text{path-connected}}$

for any two points $z, w \in D$ there is a curve $\gamma: [a,b] \to D$ with $\gamma(a) = z$ and $\gamma(b) = w$



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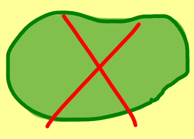
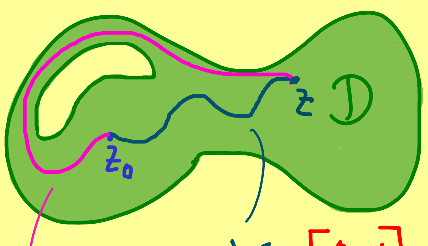


If $\oint_\gamma f(z)\,dz = 0$ for all closed curves γ, then f has an antiderivative.

Proof:



$\tilde{\gamma}_z$

$\gamma_z: [0,1] \to D$

$\gamma_z(0) = z_0, \quad \gamma_z(1) = z$

For $z_0, z \in D$, define:

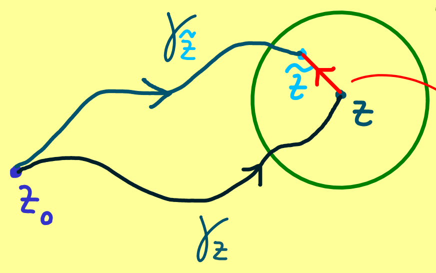
$$F(z) := \int_{\gamma_z} f(\zeta)\,d\zeta$$

well-defined!

$\Rightarrow$ $\tilde{\gamma}_z^- + \gamma_z$ closed curve: $0 = \oint_{\tilde{\gamma}_z^- + \gamma_z} f(\zeta)\,d\zeta = \int_{\tilde{\gamma}_z^-} f(\zeta)\,d\zeta + \int_{\gamma_z} f(\zeta)\,d\zeta$

$$\Longrightarrow \int_{\gamma_z} f(\zeta)\,d\zeta = \int_{\tilde{\gamma}_z} f(\zeta)\,d\zeta$$

Show: $F' = f$



$B_\varepsilon(z)$

$\gamma_{z\tilde{z}}$ line connecting $\tilde{z}$ with z

$$\left| \frac{F(\tilde{z}) - F(z)}{\tilde{z} - z} - f(z) \right| = \left| \frac{F(\tilde{z}) - F(z) - f(z)(\tilde{z} - z)}{\tilde{z} - z} \right|$$

$$= \frac{1}{|\tilde{z}-z|} \cdot \left| \int_{\gamma_{\tilde{z}}} f(\zeta)\,d\zeta - \int_{\gamma_z} f(\zeta)\,d\zeta - f(z)(\tilde{z}-z) \right|$$

$$= \frac{1}{|\tilde{z}-z|} \cdot \left| \int_{\gamma_{z\tilde{z}}} f(\zeta)\,d\zeta - \int_{\gamma_{z\tilde{z}}} f(z)\,d\zeta \right| = \frac{1}{|\tilde{z}-z|} \cdot \left| \int_{\gamma_{z\tilde{z}}} \big(f(\zeta) - f(z)\big)\,d\zeta \right|$$

$$\leq \frac{1}{\cancel{|\tilde{z}-z|}} \max_{\zeta \in \mathrm{Ran}(\gamma_{z\tilde{z}})} |f(\zeta) - f(z)| \cdot \cancel{\mathrm{length}}(\gamma_{z\tilde{z}}) \leq \max_{\zeta \in B_\varepsilon(z)} |f(\zeta) - f(z)| \xrightarrow{\varepsilon \to 0} 0$$