BECOME A MEMBER
ON STEADY

The Bright Side of
Mathematics

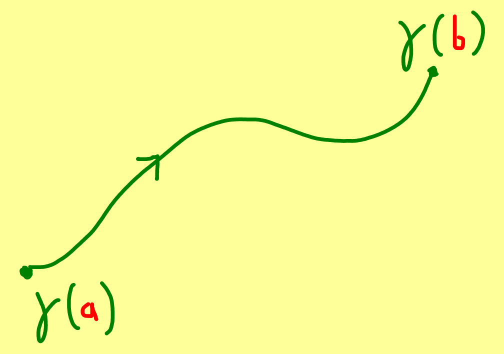
# Complex Analysis – Part 20

**Definition:** $U \subseteq \mathbb{C}$ open , $f: U \to \mathbb{C}$.

$F: U \to \mathbb{C}$ is called a primitive/antiderivative of $f$

if $F' = f$. complex derivative!

**Fact:** If $f: U \to \mathbb{C}$ has an antiderivative $F: U \to \mathbb{C}$, then:




$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

parametrized curve $\gamma: [a,b] \to U$

**Proof:**
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt ,$$
$$= \int_a^b \frac{d}{dt}(F \circ \gamma)(t)\,dt$$

$$\frac{d}{dt}(F \circ \gamma)(t) = F'(\gamma(t)) \cdot \gamma'(t) \quad \text{chain rule}$$

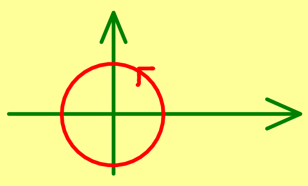
fundamental theorem of calculus
$$= (F \circ \gamma)(t)\Big|_a^b = F(\gamma(b)) - F(\gamma(a))$$

**Corollary:** If $f: U \to \mathbb{C}$ has an antiderivative and $\gamma$ is closed, then:

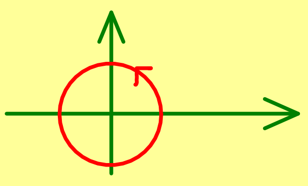
$$\oint_\gamma f(z)\,dz = 0$$

**Example:** (a) $U = \mathbb{C}\setminus\{0\}$ , $f(z) = \frac{1}{z^2}$ antiderivative: $F(z) = -\frac{1}{z}$

$$\Rightarrow \oint_\gamma f(z)\,dz = 0$$



(b) $U = \mathbb{C}\setminus\{0\}$ , $f(z) = \frac{1}{z}$



We know: $\oint_\gamma f(z)\,dz = 2\pi i$ with $\gamma: [0,2\pi] \to U$ , $\gamma(t) = e^{it}$

$\Rightarrow$ no antiderivative for $\frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$

(c) $U = D_{-\pi}$

$\log: D_{-\pi} \to \mathbb{C}$

$\log'(z) = \frac{1}{z}$ $\Rightarrow \oint_\gamma f(z)\,dz = 0$