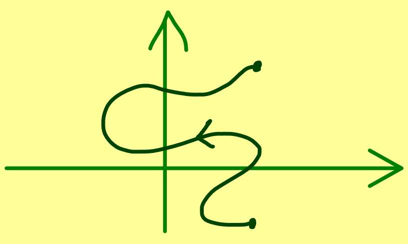
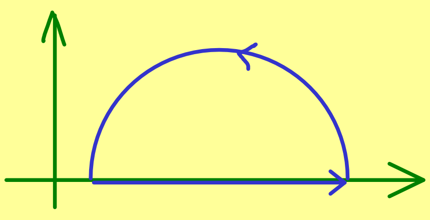
The Bright Side of Mathematics

Complex Analysis – Part 19

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$



$\gamma : [a,b] \longrightarrow \mathbb{C}$ continuously differentiable



We can extend this: $\gamma : [a,b] \longrightarrow \mathbb{C}$ piecewise continuously differentiable

there are $\underset{= a}{a_1}, a_2, a_3, \ldots, \underset{= b}{a_{n+1}} \in [a,b]$

such that $\gamma|_{[a_i, a_{i+1}]}$ is continuously differentiable

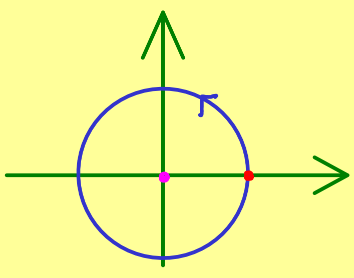
define: $$\int_\gamma f(z)\,dz := \sum_{i=1}^{n} \int_{\gamma|_{[a_i,a_{i+1}]}} f(z)\,dz$$

If $\gamma(a) = \gamma(b)$, then γ is called a closed curve and we write:

$$\oint_\gamma f(z)\,dz$$

Important example: $\oint_\gamma \frac{1}{z}\,dz$, $\quad \gamma : [0,2\pi] \longrightarrow \mathbb{C}$, $\quad t \longmapsto e^{it}$

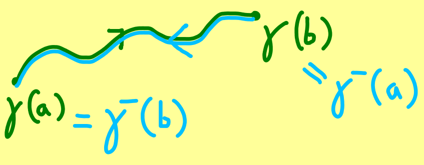
$$= \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\cdot i\,\cancel{e^{it}}\,dt = 2\pi\cdot i$$



Properties: $f, g : U \longrightarrow \mathbb{C}$ continuous, $\gamma : [a,b] \longrightarrow \mathbb{C}$ piecewise continuously differentiable

(a) $$\int_\gamma \big(\alpha\cdot f(z) + \beta\cdot g(z)\big)\,dz = \alpha\cdot\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz \quad \text{for all } \alpha,\beta\in\mathbb{C}$$

(b) If γ^- is γ with reverse orientation, $\left(\gamma^-(t) := \gamma(-t+a+b)\right)$



then $$\int_{\gamma^-} f(z)\,dz = -\int_\gamma f(z)\,dz$$

(c) $$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt\right| \leq \int_a^b \left|f(\gamma(t))\cdot\gamma'(t)\right|dt$$

$$= \int_a^b \left|f(\gamma(t))\right|\cdot\left|\gamma'(t)\right|dt \leq \sup_{z\in \mathrm{Ran}(\gamma)}\left|f(z)\right|\cdot\int_a^b\left|\gamma'(t)\right|dt$$

$$= \max_{z\in\mathrm{Ran}(\gamma)}\left|f(z)\right|\cdot \operatorname{length}(\gamma)$$