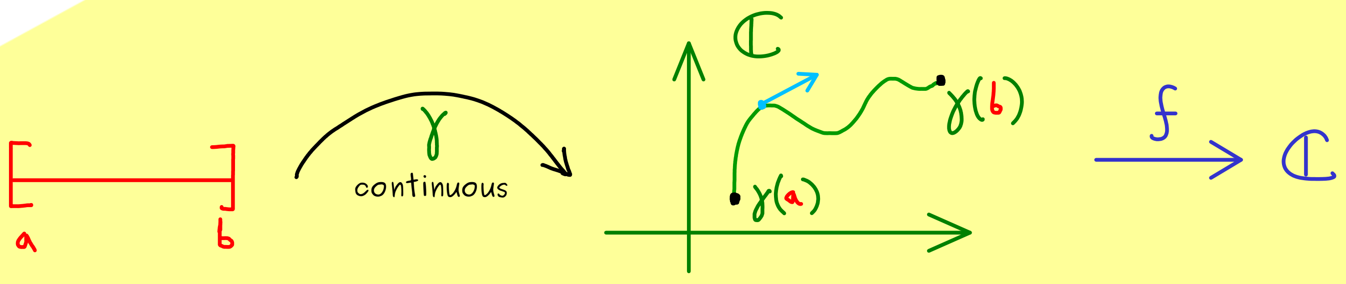
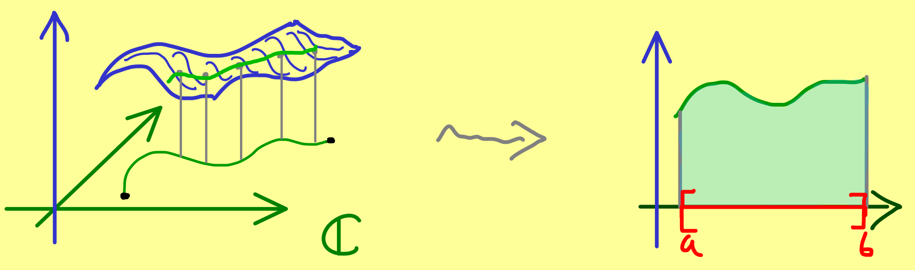
The Bright Side of Mathematics

Complex Analysis – Part 18



First look at $f: \mathbb{C} \longrightarrow \mathbb{R}$

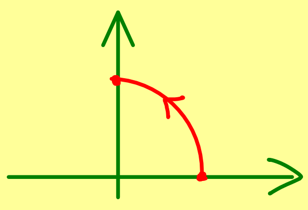


<u>Definition:</u> For a <u>parametrized curve</u> $\gamma: [a,b] \longrightarrow \mathbb{C}$ continuously differentiable with $\gamma': [a,b] \longrightarrow \mathbb{C}$, we define:

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$

for <u>continuous</u> functions $f: U \longrightarrow \mathbb{C}$ with $\mathrm{Ran}(\gamma) \subseteq U$.

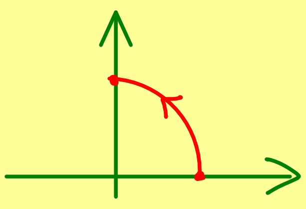
<u>Examples:</u> (a) $f(z) = z$, $\gamma_1: [0, \frac{\pi}{2}] \longrightarrow \mathbb{C}$, $t \longmapsto e^{it}$



$$\int_{\gamma_1} f(z)\, dz = \int_0^{\frac{\pi}{2}} f(\underbrace{\gamma_1(t)}_{e^{it}}) \cdot \underbrace{\gamma_1'(t)}_{i \cdot e^{it}}\, dt = i \cdot \int_0^{\frac{\pi}{2}} e^{2it}\, dt = i \cdot \frac{1}{2i} e^{2it} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \cdot \left(e^{i\pi} - 1\right) = -1$$

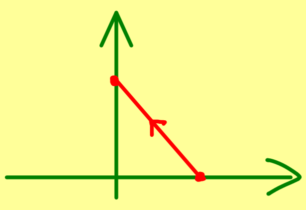
(b) $f(z) = z$, $\gamma_2: [0,1] \longrightarrow \mathbb{C}$, $t \longmapsto e^{i\frac{\pi}{2}t}$



$$\int_{\gamma_2} f(z)\, dz = \int_0^1 f(\underbrace{\gamma_2(t)}_{e^{i\frac{\pi}{2}t}}) \cdot \underbrace{\gamma_2'(t)}_{i\frac{\pi}{2}e^{i\frac{\pi}{2}t}}\, dt = i \cdot \frac{\pi}{2} \int_0^1 e^{i\pi t}\, dt = i\frac{\pi}{2} \frac{1}{i\pi} e^{i\pi t} \Big|_0^1$$

$$= -1$$

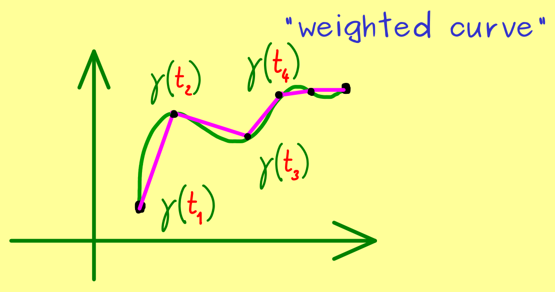
(c) $f(z) = z$, $\gamma_3: [0,1] \longrightarrow \mathbb{C}$, $t \longmapsto (1-t) + i \cdot t$



$$\int_{\gamma_3} f(z)\, dz = \int_0^1 f(\gamma_3(t)) \cdot \underbrace{\gamma_3'(t)}_{-1+i}\, dt = (-1+i) \int_0^1 \left(1 + (i-1)t\right) dt$$

$$= (-1+i)\left(t + \frac{1}{2}(i-1)t^2\right)\Big|_0^1 = (-1+i)\left(1 + \frac{1}{2}(i-1)\right) = -1$$

<u>Another visualisation:</u>



$$\sum_{i=1}^{n} f(\gamma(t_i)) \cdot \left(\gamma(t_{i+1}) - \gamma(t_i)\right)$$

$$= \sum_{i=1}^{n} f(\gamma(t_i)) \frac{\gamma(t_{i+1}) - \gamma(t_i)}{t_{i+1} - t_i} (t_{i+1} - t_i)$$

$$\xrightarrow[\text{(in some sense)}]{n \to \infty} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\, dt$$